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FOUNDATIONS, THEORY OF SETS, LOGIC

Wilson, N. L. Existence assumptions and contingent meaningfulness. Mind 65 (1956), 336–345.

This paper explores the logical and semantic status of the existence assumptions (e.g., '$(\exists x)(Fx \vee \sim Fx)$' or '$(\exists x)(x=x)$') of logistic systems such as that of "Principia Mathematica". The author argues that such assumptions, although they could not be false, would be rendered meaningless by a complete absence of (concrete) individuals. Such assumptions, if meaningful, are necessarily (analytically) true; but they are only contingently meaningful, since the world might have been such as to leave the range of their individual variables empty, and "a variable is meaningful if and only if its range is non-empty" (p. 340).

The author goes on to argue that not only formal logistic systems, but languages generally, are incapable of "expressing the (false) [but not self-contradictory] proposition that no individuals exist, or of ··· recording the fact that there are individuals" (p. 342). Such linguistic inexpressibles he calls "proto-facts" (p. 345). They limit the scope of language by imposing upon it a "descriptive incompleteness" analogous to the "logical incompleteness" of formal systems asserted by Gödel's theorem. "Language cannot synthetically record the existence of individuals because the existence of individuals is a condition of its own existence" (p. 345).

Mr. Wilson calls his own position "unabashedly platonistic" in the sense that it construes predicates as names of (real) properties. "Meaning" thus becomes "designation"; individual constants and predicates "mean" by "naming". At this point in the argument philosophic difficulties arise which might ultimately cast some doubt upon the author's central thesis. But, given an adequate theory of meaning, the general notion of "contingent meaningfulness" (for which Mr. Wilson expresses his indebtedness to Rulon Wells and to F. P. Ramsey) seems fruitfully relevant to the problem of the status of existence assumptions in logical and mathematical systems. *G. L. Kline* (New York, N.Y.).

Kurepa, George. The principle of complete induction. Bull. Internat. Acad. Yougoslave Sci. Beaux-Arts (N.S.) 6 (1952), 97–103.

A translation of the article reviewed in MR 14, 938.

Celtin, G. S. Associative calculus with insoluble equivalence problem. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 370–371. (Russian)

An associative system with an unsolvable word problem having a relatively very moderate number of defining relations is given. This is the following system Z: the alphabet consists of five letters (a, b, c, d, e) and the seven defining relations are $ac \leftrightarrow ca$, $ad \leftrightarrow da$, $bc \leftrightarrow cb$, $bd \leftrightarrow db$, $abac \leftrightarrow abace$, $eca \leftrightarrow ae$, $edb \leftrightarrow be$. An associative system is called to be of special kind if all the defining relations have an empty right hand side. The word problem for every system of special kind can be reduced to that of the system Z. [The unsolvability of the word problem for systems of special kind follows for instance from the results of P. S. Novikov concerning the unsolvability of the word problem for groups [see Trudy Mat. Inst. Steklov. 44 (1955); MR 17, 706]. Construct the system Z with the same alphabet as Z adding to the seven relations above the two new relations $ccca \leftrightarrow ccc$, $cccb \leftrightarrow ccc$. In this system the problem $X \leftrightarrow ccc$ is undecidable. No details of the proofs are given. *S. Schwarz* (Bratislava).

★ **Davis, M. D. A note on universal Turing machines.** Automata studies, pp. 167–175. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

The author defines a universal Turing machine as one for which the set of Gödel numbers of the instantaneous descriptions is complete. By instantaneous descriptions he means initial tapes and states leading to a terminating machine reaction, and by complete he means a recursively enumerable set of integers into which every recursively enumerable set may be mapped by a suitable recursive function.

This definition of a universal machine is justified by showing that 1) any computation may be encoded for such a machine, 2) any machine which may be encoded to produce any recursively enumerable set is such a machine, and 3) the encoding in 1) may be achieved by a machine whose set of Gödel numbers for the instantaneous descriptions is not merely not complete but is actually recursively enumerable. *S. Gorn.*

★ **Shannon, Claude E. A universal Turing machine with two internal states.** Automata studies, pp. 157–165. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

The author shows that any Turing machine with an alphabet of m letters and n states is 'essentially equivalent' to a machine with $4mn+m$ symbols and two states. Thus a universal Turing machine may be constructed with only two states.

It is also shown that a single state machine cannot compute the successive digits of an irrational number. Thus a universal machine must have more than one state.

Finally it is shown that any machine with n states and an alphabet of m letters is equivalent to a machine with an alphabet of two letters and $(2^l-1)n$ states, where $l=1+[\log_2 m]$. *S. Gorn* (Philadelphia, Pa.).

★ **McCarthy, John. The inversion of functions defined by Turing machines.** Automata studies, pp. 177–181. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

This is an informal discussion of the problem of designing a machine to solve "well-defined" intellectual problems. The author identifies this problem with that of inverting a partial recursive function $f_m(n)$ to yield a function $g(m, r)$ such that $f_m(g(m, r))=r$. *S. Gorn.*

★ **de Leeuw, K.; Moore, E. F.; Shannon, C. E.; and Shapiro, N. Computability by probabilistic machines.** Automata studies, pp. 183–212. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

The author consider machines fulfilling the following conditions. 1. For every input composed of a finite binary sequence there is an output composed of a finite sequence of symbols chosen from a set S. 2. The functional relationship between the input and output sequences is computable. 3. The output corresponding to an initial segment of any given input will be an initial segment of the output corresponding to that given input. For such machines, then, there is a computable function $f(A)$ whose range is a subset of the set of infinite sequences chosen from S, and whose domain is the set of infinite binary sequences A. Any such machine together with a fixed input sequence A is called an A-machine. The set of output symbols, S, of an A-machine is called A-enumerable. The recursively enumerable sets are those which are A-enumerable for A the sequence containing 1 exclusively; the authors also call them 1-enumerable, and the enumerating machine a 1-machine. If the input tape is produced by a random device which yields the bits independently and with probability p of yielding a 1, the combination of the random device and the machine is called a p-machine. S is called strongly p-enumerable if there is a p-machine producing S in any order with non-zero probability. If S_m is the set of output symbols printed by a p-machine with a probability greater than $\frac{1}{2}$, then S_m is called p-enumerable. Let A_p be the binary expansion of p, where $0<p<1$. Then the authors show that the concepts 'p-enumerable', 'strongly p-enumerable', and 'A_p-enumerable' are effectively equivalent. Furthermore, if p is computable, any p-enumerable or strongly p-enumerable set is 1-enumerable, whereas if p is not computable there are p-enumerable and strongly p-enumerable sets which are not 1-enumerable. Because of condition 3 above, these concepts and results extend readily to the case where S is a fixed sequence of symbols. By a 'stochastic machine' the authors mean one with an initial state X_0, a countable sequence of states $X_1, X_2, \cdots$, a countable set of output symbols $s_1, s_2, \cdots$ and a (not necessarily computable) transition probability function,

$$P(X_{j_1}, \cdots, X_{j_u}; X_p; S_q)$$

giving the probability that the machine, having gone through states X_{j_1} to X_{j_u}, will print s_q and go to state X_p. If P is computable, the machine is called a computable stochastic or c-s-machine. To any p-machine M one may associate an "equivalent" stochastic machine. It is a c-s-machine if and only if p is computable. The authors show, finally, that every c-s-machine is effectively equivalent to a $\frac{1}{2}$-machine, and every c-s-enumerable set is effectively 1-enumerable. *S. Gorn.*

Péter, Rózsa. Die beschränkt-rekursiven Funktionen und die Ackermannsche Majorisierungsmethode. Publ. Math. Debrecen **4** (1956), 362–375.

The author mentions an example due to Ackermann [Math. Ann. **99** (1928), 118–133], of a general recursive function which is not bounded by any primitive recursive function. An example, due to Skolem [Norske Vid. Selsk. Forh. **17** (1944), 103–106; MR **8**, 4], of a general recursive characteristic function which is not primitive recursive shows that this unboundedness is not an essential property of recursive functions which are not primitive recursive. However, in the construction of Skolem's example, an intermediate function is defined which is not bounded by any primitive recursive function, and this situation the author shows to hold always. If, in a system of equations defining a function from a set of initial primitive recursive functions, all functions occurring (auxiliary functions as well as the defined function) are bounded by primitive recursive functions, then the defined function is primitive recursive.

H. G. Rice (Pittsburgh, Pa.).

Hu, Shih-Hua. On the primitive recursiveness of certain recursions. Acta Math. Sinica **6** (1956), 93–104. (Chinese. English summary)

Rényi, Alfred. The ideological significance of the geometry of Bolyai-Lobačevskiĭ. Časopis Pěst. Mat. **78** (1953), 149–168. (Czech)

★ **Yiftah, Shimon. Constantes fondamentales des théories physiques.** Gauthier-Villars, Paris, 1956. xii+124 pp. 2.300 francs.

The author has collected and reported on a good deal of the recent theoretical literature concerning the following six physical constants: h the quantum of action, c the velocity of light, γ the newtonian constant of gravitation, m the mass of a nucleon, m_e the mass of an electron, and e the charge of an electron. After an introductory Chapter I, Chapter II is concerned with classifying various modern physical theories in terms of the constants entering them. For example the relativistic wave mechanics of Dirac is said to be characterized by the fact that it involves h, c, and m_e (or m). The nature of these theories is briefly outlined. In Chapter III the author introduces various dimensionless combinations of the six constants listed above and reviews the remarks of Dirac and Jordan concerning them. In particular he discusses the possibility of these constants varying with time. Chapter IV is mainly devoted to an uncritical review of Eddington's work on the determination of an equation relating the masses of the proton and electron. Chapter V is devoted to the fine structure constant taken to be equal to 137. Eddington's work is reported. The author also observes that the masses of some mesons are approximately equal to an integer times 137 times the mass of an electron. The final Chapter VI is devoted to a brief summary of cosmological theories.

A. H. Taub.

See also: Lyndon, p. 106; Malcev, p. 107; Ledley, p. 155; Bopp, p. 173.

ALGEBRA

Combinatorial Analysis

Ryser, H. J. Maximal determinants in combinatorial investigations. Canad. J. Math. **8** (1956), 245–249.

Let v be an integer, $v>1$, and let Q be a matrix of 0's and 1's, of order v. The author is interested in upper bounds for $|\det Q|$ (the absolute value of the determinant of Q) of such a nature that the bound is attained if and only if Q is an incidence matrix of a symmetric balanced incomplete block design. He gives a counter-example which suggests that one must specify the number, t, of 1's in Q and impose suitable inequalities relating t and v. As one example of suitable inequalities, define $k=t/v$, $\lambda=k(k-1)/(v-1)$ and assume that $0<\lambda<k<v$. The author's theorem for this case is as follows: $|\det Q|\leq k(k-\lambda)^{\frac{1}{2}(v-1)}$; and equality holds if and only if k, λ are integers and Q is an incidence matrix of a v, k, λ configuration. The methods of proofs (and the above-mentioned counter-example) are adapted from the theory of Hadamard matrices. *R. H. Bruck.*

★**Shestakov, V. I. On the transformation of a monocyclic sequence into a recurrent.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

A translation of the article reviewed in MR **16**, 785.

Hussain, Q. M. An alternative proof of the number of m-flats in N-dimensional finite projective geometry formed from Galois Field G.F. (p^n), where p is a prime number and n a positive integer. Proc. Pakistan Statist. Assoc. **3-4** (1954–1955), 1–2.

The properties of balanced incomplete block designs are utilized to derive the number of m-dimensional subspaces contained in a P.G. (N, p^n). *H. B. Mann.*

See also: Hall and Paige, p. 109.

Linear Algebra

★**Thurston, H. A. The number-system.** Interscience Publishers Inc., New York, 1956. viii+134 pp. $2.50.

This book has a dual purpose, to give a detailed treatment of the successive extensions of the number system and to introduce the student to the methods of abstract algebra. The first part is informal and serves as a motivation for the rigorous treatment in the second part. Starting with the whole numbers the author discusses the reasons why we wish to enlarge our number system and hints at how this can actually be effected. In the second part he carries out these constructions. Starting from the Peano axioms for whole numbers he justifies the recursive definitions of addition and multiplication and proves the basic laws of arithmetic. He then defines a hemigroup (commutative semigroup with identity satisfying the cancellation law) and commutative group, and proves that every hemigroup can be embedded in a commutative group. This paves the way for the construction of the integers and the rational numbers. The notions of a field and an ordered field are introduced, and the rational numbers are shown to form an ordered field. The real numbers are constructed by means of Cauchy sequences, and the complex numbers are defined as ordered pairs of real numbers. *B. Jónsson* (Minneapolis, Minn.).

★**Cahen, Gilbert. Eléments de calcul matriciel.** Dunod, Paris, 1955. vi+94 pp.

A concise theoretical exposition with applications to vibration and to electric circuits, including the Kron method.

Roth, William E. On the characteristic polynomial of the product of several matrices. Proc. Amer. Math. Soc. **7** (1956), 578–582.

Let A be an $n\times n$ matrix; let $A_i=A+D_i$, where D_i is a matrix of rank 1 of the special type $D_i=R^*S_i$ ($i=1, \cdots, r$; R invariable). The characteristic polynomial of $\prod A_i$ is given explicitly in terms of the coefficients of the characteristic polynomials of the A_i. In particular, $\det(xI-A^r)$ can be written in terms of the coefficients of $\det(xI-A)$. *J. L. Brenner* (Palo Alto, Calif.).

Ville, J. A. De la transposition. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 137–154 (1956).

One of the problems treated in this paper can be paraphrased as follows. Let $\Psi=[\psi_1, \cdots, \psi_n]$ be a basis of a vector-space over reals; we wish to define a "transpose" Ψ^T of Ψ with the properties $\Psi^{TT}=\Psi$; $(\Phi\Psi)^T=\Psi^T\Phi^T$; here Φ and Ψ^T are interpreted as sets of vectors in the dual space. Thus $\Phi\Psi$ is a matrix A of reals, and we require further that $(\Phi\Psi)^T$ be the transpose of A in the usual sense. The solution of the problem is essentially the conclusion that $\Psi^T\Psi$ be symmetric. {For a more fruitful generalization of the notion of transposition see Katz and Olkin [Duke Math. J. **20** (1953), 331–337; MR **14**, 939].} *J. L. Brenner* (Palo Alto, Calif.).

Cherubino, Salvatore. Su una disuguaglianza in matrici. Boll. Un. Mat. Ital. (3) **11** (1956), 126–132.

If each component of the row vector $x=(x_1, \cdots, x_n)$ is real and positive we write $x>0$. If A is a given real matrix with n rows and p columns, the question arises whether the inequalities $xA>0$, $x>0$ are compatible. Sufficient, and in some cases necessary, conditions for this are obtained (i) when A is a constant matrix and (ii) when the elements of A are bounded functions. *D. E. Rutherford* (St. Andrews).

Cherubino, Salvatore. Su un'equazione della teoria delle vibrazioni. Boll. Un. Mat. Ital. (3) **11** (1956), 133–136.

The simultaneous reduction to diagonal form of two Hermitian matrices A, B, where A is positive definite, is obtained by a variation of the usual method. New bounds for the roots of $|A\lambda+B|=0$ are derived. *D. E. Rutherford* (St. Andrews).

Amir-Moéz, Ali R. Extreme properties of eigenvalues of a hermitian transformation and singular values of the sum and product of linear transformations. Duke Math. J. **23** (1956), 463–476.

The singular values of a matrix A are the non-negative square roots of the eigenvalues of AA^*. For two hermitian matrices A, B several relations are known between the eigenvalues of A, B and those of AB or $A+B$. For the corresponding problem for general matrices the singular values have been mainly studied so far [see, e.g., A. Horn, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 374–375; MR **13**, 565]. The results of this paper are generalizations of the following results of Weyl [Math. Ann. **71** (1912), 441–479], Lidskiĭ [Dokl. Akad. Nauk. SSSR (N.S.) **75** (1950), 769–

772; MR 12, 581], and Wielandt [Proc. Amer. Math. Soc. 6 (1955), 106–110; MR 16, 785]. Let A, B be hermitian and $C=A+B$ with eigenvalues α_i, β_i, γ_i arranged in non-increasing order. Then Weyl's theorem is: $\gamma_{i+j-1}\leq\alpha_i+\beta_j$ for $i+j\leq n+1$ and $\gamma_{i+j-n}\geq\alpha_i+\beta_j$ for $i+j\geq n+1$. Lidskiĭ's theorem is: $\gamma_{i_1}+\cdots+\gamma_{i_k}-\beta_{i_1}-\cdots-\beta_{i_k}\leq\alpha_1+\cdots+\alpha_k$ for every sequence of integers $i_1, \cdots, i_k$ with $1\leq i_1<\cdots<i_k\leq n$. Wielandt found a new proof for Lidskiĭ's theorem based on the following relations, generalizing Courant's classical max min theorem: Let H be hermitian with eigenvalues $h_1\geq\cdots\geq h_n$. Let $i_1, \cdots, i_k$ be integers such that $1\leq i_1<\cdots<i_k\leq n$. Then

$$h_{i_1}+\cdots+h_{i_k}=\sup\inf[(Hx_1, x_1)+\cdots+(Hx_k, x_k)],$$

where inf is taken over an orthonormal set of vectors $x_1, \cdots, x_k$ where $x_p \in M_p$, a space of dimension i_p ($p=1, \cdots, k$), and sup is taken over all spaces $M_1\subset\cdots\subset M_k$. The generalization of this max min theorem to any symmetric function $\varphi(t_1, \cdots, t_k)$, non-decreasing in each variable, instead of $t_1+\cdots+t_k$ is then used to find the main inequalities. A generalization is proved not only for the case $1\leq i_1<\cdots<i_k\leq n$, but also for the case $i_1\leq\cdots\leq i_k$, where $i_p\geq p$ and the case when $i_p\leq n-k+p$, $p=1, \cdots, k$.

For Hermitian matrices A, B the following generalization of the theorems of Weyl, Lidskiĭ and Wielandt is found: let $i_1\leq\cdots\leq i_k$ and $j_1\leq\cdots\leq j_k$ be such that $i_k\leq n$, $j_k\leq n$, $i_p+j_p\geq n+p$, $p=1, \cdots, k\leq n$. Then

$$\alpha_{i_1'}+\cdots+\alpha_{i_k'}+\beta_{j_1'}+\cdots+\beta_{j_k'}\leq\gamma_{(i_1+j_1-n)'}+\cdots+\gamma_{(i_k+j_k-n)'}$$

for certain sequences $i_1', \cdots, i_k'$ which include the ones defined in the following way. The numbers i_p' are strictly increasing and $i_p'\leq i_p$, further $j_p\leq i_p'$ for all strictly increasing sequences $\{j_p\}$ for which $j_p\leq i_p$. If $i_p+j_p\leq n-k+p+1$, $p=1, \cdots, k\leq n$, then

$$\alpha_{i_1''}+\cdots+\alpha_{i_k''}+\beta_{j_1''}+\cdots+\beta_{j_k''}\geq$$
$$\gamma_{(i_1+j_1-1)''}+\cdots+\gamma_{(i_k+j_k-1)''}$$

for certain sequences $i_1'', \cdots, i_k''$, e.g. the smallest strictly increasing sequences $\{i_p''\}$ for which $i_p''>i_p$. If the sequences $i_1, \cdots, i_k$ and $j_1, \cdots, j_k$ are strictly increasing then only Weyl's theorem is obtained. It is pointed out that the Lidskiĭ-Wielandt theorem covers more cases than Weyl's, but the above theorems cover even more cases. The results obtained for hermitian matrices are then applied to yield inequalities for the singular values of sums and products of arbitrary matrices. If A and B are non-negative hermitian then AB is known to have real eigenvalues and analogous inequalities for the eigenvalues themselves of AB are obtained. (The paper contains a number of printing errors, e.g. on p. 469, line 8 from bottom: replace 2.8(2) by 2.6). *O. Taussky-Todd.*

Morozov, V. V. On the equation of fifth degree. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 29–39. (Russian)

This work establishes an effective formula for transforming the general equation of the fifth degree into the one-parametric form of Klein and observing the behavior of the critical manifold [of multiple roots] under this transformation. (Author's Summary.) *Harvey Cohn.*

Weaver, Milo. Zero integral homogeneous symmetric functions in certain rings. Amer. Math. Monthly **63** (1956), 387–391.

The author considers integral homogeneous symmetric functions of degree n over a ring. A generalisation of Hensel's theorem is then proved for such functions. The following analogue of Wolstenholme's theorem occurs as a special case of a corollary of the author's theorem:

$$\sum_{r=1}^{p-1} r^{p(p^2-p-1)}\equiv 0 \pmod{p^2},$$

p being an odd prime. Wolstenholme's theorem gives

$$\sum_{r=1}^{p-1} r^{p^2-p-1}\equiv 0 \pmod{p^2},\ p>3.$$

H. Gupta (Hoshiarpur).

Dionísio, J. J. Some remarks on Galois theory. Rev. Fac. Ci. Univ. Coimbra **24** (1955), 12–17.

The author gives an exposition of some standard results of classical Galois theory using the familiar methods of linear algebra. *W. Ledermann* (Manchester).

See also: Szekeres, p. 121; Fraeys de Veubeke, p. 154; Laha, p. 160; Belevitch, p. 171; Oono, p. 171; Lieberman, p. 180.

Partial Order Structures

Jakubík, Ján. Direct decompositions of the unity in modular lattices. Czechoslovak Math. J. **5(80)** (1955), 399–411. (Russian. English summary)

The author is concerned with the existence of a common refinement for two decompositions $1=\mathbf{U}a_i=\mathbf{U}b_j$ in a complete modular lattice. The background for the investigation can be found in Kuroš's "Theory of groups" [2nd ed., Gostehizdat, Moscow, 1953, Ch. XI; MR 15, 501]. Kuroš assumed a certain strengthening of the modular law and proved that a necessary and sufficient condition for a common refinement is $1\bar{\theta}_i\phi_j\theta_i=0$ for all i and j, where θ_i, ϕ_j are the projections associated with the decompositions and $\bar{\theta}_i$ is the projection complementary to θ_i. The author assumes only completeness and modularity and gives the following necessary and sufficient condition: that the complete sublattice generated by the a's and b's be distributive. He also derives Kuroš's result with just the ordinary modular law. A number of related results are given and applications are made to "algebraic systems" in the sense of Malcev [Mat. Sb. N.S. **35(77)** (1954), 3–20; MR 16, 440]. *I. Kaplansky.*

Kurepa, Georges. Le problème de la mesure et les transformations monotones des ensembles partiellement ordonnés. Bull. Internat. Acad. Yougoslave Sci. Beaux-Arts (N.S.) **6** (1952), 91–96.

A translation of the article reviewed in MR **14**, 960.

Lyndon, Roger C. The representation of relation algebras. II. Ann. of Math. (2) **63** (1956), 294–307.

A result by A. Tarski [Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 572–581, 582–588; **58** (1955), 56–64; MR 16, 554] states that there exists a set of equations characterizing the class of all those relation algebras which are isomorphic to algebras of relations. In this paper such a set of equations is explicitly obtained. The equations, which are infinite in number and are constructed recursively, are quite involved and the reviewer has difficulties following some of the steps in the proof of their sufficiency. Tarski's result contradicts earlier conclusions by the author [Ann. of Math. (2) **51** (1950), 707–729; MR 12, 237]. The flaw in that paper was

located by Dana Scott. It consists in the assumption that in a complete atomistic set-field every set is the union of atoms. The necessary changes in the theorems are indicated. In particular, Theorem IV used to read: "The class of all relational algebras that are isomorphic to proper relational algebras cannot be characterized by any set of algebraic axioms." This is changed to: "The class of all strongly representable atomistic relation algebras is not definable by universal sentences." Here an atomistic relation algebra is called strongly representable if it is isomorphic to a proper relation algebra in which the universal element is the set-theoretic union of atoms.

B. Jónsson (Minneapolis, Minn.).

Postley, J. A. A method for the evaluation of a system of Boolean algebraic equations. Math. Tables Aids Comput. **9** (1955), 5–8.

The author describes a way of using punched cards to assist in making certain calculations involving a fairly large number of variables which are always either 0 or 1. The basic principle is that if one indicates the fulfilment of a condition in a given instance by punching a hole in a definite position in a card for that instance, then the simultaneous fulfillment of the condition in all the instances can be tested by sighting through the batch of cards. This idea is applied to calculating the y_r from the x_r when we have equations of the form

$$y_r=\phi(x_r, f_r(x_1, \cdots, x_n), g_r(x_1, \cdots, x_n)),$$

where ϕ, f_r, g_r are Boolean functions in alternative (disjunctive) normal form. *H. B. Curry.*

See also: Fleischer, p. 136; Hofmann, p. 141; Schröder, p. 152; Ledley, p. 155; Zemmer, p. 180.

Rings, Fields, Algebras

Malcev, A. I. Quasiprimitive classes of abstract algebras. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 187–189. (Russian)

If the algebras of a class satisfy a fixed system of conditional equations, the class is called quasiprimitive. Each of the following is necessary and sufficient that a class K be quasiprimitive: (1) K is finitely free and locally defined; (2) K is locally defined and the direct product of algebras from K belongs to K. The theorem of Evans [J. London Math. Soc. **28** (1953), 76–80; MR **14**, 839] is extended to quasiprimitive classes. *R. A. Good.*

Lamprecht, Erich. Bewertungssysteme und Zetafunktionen algebraischer Funktionenkörper. I. Math. Ann. **131** (1956), 313–335.

Let A be a function field of 2 variables over a finite field of constants k. Consider a pair of subfields $K\neq K'$ of A, each of transcendence degree 1 over k, and each algebraically closed in A. Let $B(K, K')$ denote the set of those places of A which induce in K' but not in K an isomorphism. For each $\mathfrak{P}\in B(K, K')$ the image field $A\mathfrak{P}$ is an algebraic function field of 1 variable over a finite field of constants; let $Z_{A\mathfrak{P}}(s)$ be the corresponding zeta function. The author considers the zeta function

$$Z_A(s; K, K')=\prod_{\mathfrak{P}\in B(K,K')} Z_{A\mathfrak{P}}(s)$$

of A belonging to the pair K, K'. [For these zeta functions, see also Lamprecht, Math. Z. **64** (1956), 47–71; MR **17**, 947.] He first shows, if also $K''\neq K$, then the two sets $B(K, K')$ and $B(K, K'')$ differ only by a finite number of places, and hence $Z_A(s; K, K')$ and $Z_A(s; K, K'')$ differ only by a finite number of factors. As a corollary we get the following statement: The zeros and poles of $Z_A(s; K, K')$ which do not lie on the lines $\Re(s)=0, \frac{1}{2}, 1$ are already determined by A and K (and do not depend on the choice of K'). As to the lines $\Re(s)=0, 1$, it is already sufficient to exclude those zeros and poles s for which $\Im(s)\neq 2\pi in/\log q$, where q denotes the number of elements in k and n an arbitrary integer. Furthermore, the difference of the orders of the zeros of $Z_A(s; K, K')$ in the points $s=0$ and $s=1$ does not depend on the choice of K'. [See also, Lang and Weil, Amer. J. Math. **76** (1954), 819–827; MR **16**, 398.] — A second result is the following: If A is a finite algebraic and pure inseparable extension of A, denote by $\bar{K}, \bar{K}'$ the algebraic closures of K, K' respectively in $\bar{A}$. Then $Z_A(s; K, K')=Z_{\bar{A}}(s; \bar{K}, \bar{K}')$. — In order to prove the above mentioned results on zeta functions, the author first develops a pure algebraic theory of "reduction of constants" which describes the connection of A/K and $A\mathfrak{P}/K\mathfrak{P}$ for almost all $\mathfrak{P}\in B(K, K')$. In this theory, k needs not to be assumed finite. The following two results may be mentioned: (i) Let A/K be separable. If A' is a subfield of A such that A/A' is algebraic, then $A\mathfrak{P}/A'\mathfrak{P}$ is algebraic for almost all $\mathfrak{P}\in B(K, K')$ and we have $[A:A']=[A\mathfrak{P}:A'\mathfrak{P}]$. (ii) If A is a purely transcendental extension of k (i.e. $A=k(x, y)$ with $x, y\in A$) then each K is a purely transcendental extension of k too (i.e. $K=k(z)$ with $z\in K$). [In connection with the latter theorem see also Chevalley, J. Math. Soc. Japan **6** (1954), 303–324, Lemma 2, p. 319; MR **16**, 672.] *P. Roquette.*

Lamprecht, Erich. Bewertungssysteme und Zetafunktionen algebraischer Funktionenkörper. II. Arch. Math. **7** (1956), 225–234.

Let A be a function field of 2 variables over its field of constants k. Consider a subfield K of A, of degree of transcendency 1 over k, and algebraically closed in A. If $K'\neq K$ is another such subfield, consider the set B of all places $\mathfrak{P}$ of A which induce in K' but not in K an isomorphism. The image field $A\mathfrak{P}$ contains $K'\mathfrak{P}$ and is therefore of degree of transcendency 1 over $K\mathfrak{P}$. The author is asking for properties which belong to A/K and $A\mathfrak{P}/K\mathfrak{P}$ for almost all (i.e. all but a finite number of) $\mathfrak{P}$ in B. For the first part of these investigations, see the paper reviewed above. The main result of this paper is as follows: Let $g(A/K)$ be the genus of A/K, and $G(A/K)$ the conservative genus, defined to be the minimum of the genera of all constant extensions of A/K. Let us assume that A/K is separable. Then, for almost all $\mathfrak{P}\in B$, $K\mathfrak{P}$ is the exact field of constants of $A\mathfrak{P}$, and we have

$$G(A/K)=G(A\mathfrak{P}/K\mathfrak{P})\leq g(A\mathfrak{P}/K\mathfrak{P})\leq g(A/K).$$

Among others, there are the following corollaries of this theorem: (1) If $G(A/K)=g(A/K)$, then $G(A\mathfrak{P}/K\mathfrak{P})=g(A\mathfrak{P}/K\mathfrak{P})$ for almost all $\mathfrak{P}\in B$. [See also Deuring, Nachr. Akad. Wiss. Göttingen. Math-Phys. Kl. IIa. **1955**, 13–42; MR **17**, 17.] (2) $g(A/K)-g(A\mathfrak{P}/K\mathfrak{P})$ is a multiple of $(p-1)/2$ for almost all $\mathfrak{P}\in B$. [See Tate, Proc. Amer. Math. Soc. **3** (1952), 400–406; MR **13**, 905.] (3) Let $\bar{g}(A/k)$ be the maximum of the genera of the subfields of A of degree of transcendency 1 over k. Then, if k is perfect, we have $\bar{g}(A/k)\leq \max(g(K/k), g(A/K))$. Furthermore, if in addition $g(K/k)>G(A/K)$ then $g(K/k)=\bar{g}(A/k)$ and K is the only field with the latter property. — The author applies these results to the case where k is a finite field.

He is asking for those properties of the zetafunction $Z_A(s; K, K')$ which only depend on A and not on K and K' (for the definition of the zetafunction, see his first part mentioned above). In particular, if A can be written as the independent compositum of two fields of degree l over k, he succeeds to construct a zetafunction of A which is determined uniquely by A alone. [See also Lamprecht, Math. Z. **64** (1955), 47–71; MR **17**, 947.] *P. Roquette* (Hamburg).

McCarthy, Paul J. Witt's cancellation theorem in valuation rings. Proc. Amer. Math. Soc. **7** (1956), 515–516.

Let K be a field with an exponential valuation V, and R the set of all elements of K whose valuation is non-negative. The units of R are those elements whose valuation is zero. Two symmetric matrices, A and B, over R are said to be congruent (written $A \cong B$) if there is a matrix T over R whose determinant is a unit in R and for which $T^TAT = B$. The author proves that if 2 is a unit in R, if A, B, and C are non-singular symmetric matrices over R, and if the direct sums $A \dotplus B$, $A \dotplus C$ are congruent, then $B \cong C$. He thus extends results of E. Witt [J. Reine Angew. Math. **176** (1936), 31–44], B. W. Jones [Bull. Amer. Math. Soc. **48** (1942), 133–142; MR **3**, 260]. and W. H. Durfee [Duke Math. J. **11** (1944), 687–697; MR **6**, 114], the last of whom proved this theorem assuming the valuation complete.

B. W. Jones (Boulder, Colo.).

Foster, Alfred L. Ideals and their structure in classes of operational algebras. Math. Z. **65** (1956), 70–75.

[For terminology see the authors papers in Math. Z. **58** (1953), 306–336; **59** (1953), 191–199; **62** (1955), 171–188; MR **15**, 194; **17**, 452.] An ideal (congruence relation) $\mathfrak{J}$ of an algebra $\mathfrak{B}$ is said to be primal if $\mathfrak{B}/\mathfrak{J}$ is a primal algebra. It is shown that if $\mathfrak{A}$ is a $\mathfrak{P}$-algebra ($\mathfrak{P}$ a primal algebra), then every ideal of $\mathfrak{A}$ is the intersection of primal ideals. Also, for any given ideal $\mathfrak{J}$ of $\mathfrak{A}$ the following three conditions are equivalent: (i) $\mathfrak{J}$ is primal, (ii) $\mathfrak{J}$ is maximal, (iii) $\mathfrak{A}/\mathfrak{J} \cong \mathfrak{P}$. Analogous results are established for $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_k\}$-algebras, where $\mathfrak{P}_1, \cdots, \mathfrak{P}_k$ are strictly independent primal algebras of a common species.

B. Jónsson (Minneapolis, Minn.).

Zemmer, Joseph L. Some remarks on p-rings and their Boolean geometry. Pacific J. Math. **6** (1956), 193–208.

A. Foster showed [Duke Math. J. **12** (1945), 143–152; MR **7**, 1] that the idempotents of a commutative ring R form a Boolean algebra B under the operations $x \cup y = x+y-xy$ and $x \cap y = xy$. He further proved [Acta Math. **84** (1951), 231–261; MR **12**, 584] that, for a fixed positive prime p, this establishes up to isomorphism a one-to-one correspondence between p-rings and Boolean algebras. The author gives a simpler proof of this result. He then observes that for a p-ring R the function $\varphi(x) = x^{p-1}$ on R onto B is a Boolean valuation in the sense that $\varphi(x) = 0$ if and only if $x = 0$, $\varphi(xy) = \varphi(x) \cap \varphi(y)$, and $\varphi(x+y) \subseteq \varphi(x) \cup \varphi(y)$. The corresponding "distance function" is defined by $d(x, y) = \varphi(x-y)$. R is said to have free mobility if every congruence between subsets of R can be extended to an isometry of R, and the principal result states that for $p > 2$ this happens if and only if the Boolean algebra B is complete. *B. Jónsson* (Minneapolis, Minn.).

McCoy, Neal H. The prime radical of a polynomial ring. Publ. Math. Debrecen **4** (1956), 161–162.

Let N, N' be the intersection of all prime ideals of the rings R and the polynomial ring $R[x]$ respectively. The author proves that $N' = N[x]$. {This result was obtained by a different approach by the reviewer [Canad. J. Math. **8** (1956), 355–361; MR **17**, 1179].} *S. A. Amitsur.*

Szendrei, J. Zur Holomorphentheorie der Ringe. Publ. Math. Debrecen **4** (1956), 450–454.

The author continues the work of Rédei [Acta Math. Acad. Sci. Hungar. **5** (1954), 169–195; MR **17**, 342] and himself [ibid. **5** (1954), 197–201; MR **17**, 342] on holomorphs of rings. As before he defines a "doppelhomothetismus" of a ring P and the ring of these. He then proves: A ring P has one and only one holomorph, and then it is commutative, if for every $\alpha \in P$ and all "doppelhomothetismen" a, b of P, $ab\alpha = \alpha ab$. Using this he sharpens a result of Rédei to: An integral domain (commutative) has one and only one holomorph and it is commutative. *I. N. Herstein* (New Haven, Conn.).

★**Samuel, P. Le lemme de Hensel. Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres.** 2e tirage multigraphié, pp. 9-01 — 9-05. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

Expository paper, giving a proof of the following very general form of Hensel's lemma. Let A be a ring filtered by the powers of a maximal ideal $\mathfrak{m}$, and supposed to be Hausdorff and complete for the corresponding topology; let E, E', F be three finitely generated A-modules, each of which is given the topology defined by the submodules $\mathfrak{m}^n E$ (resp. $\mathfrak{m}^n E'$, $\mathfrak{m}^n F$); F is supposed to be Hausdorff for that topology. Consider now a bilinear mapping f of $E \times E'$ into F, and let $\bar{f}$ be the corresponding bilinear mapping of $(E/\mathfrak{m}E) \times (E'/\mathfrak{m}E')$ into $F/\mathfrak{m}F$; let $y \in F$, $\alpha \in E/\mathfrak{m}E$, $\alpha' \in E'/\mathfrak{m}E'$ be such that: 1) the class $\bar{y}$ of y in $F/\mathfrak{m}F$ is $\bar{f}(\alpha, \alpha')$; 2) $F/\mathfrak{m}F$ is the sum of $\bar{f}(\alpha, E'/\mathfrak{m}E')$ and of $\bar{f}(E/\mathfrak{m}E, \alpha')$. Then there exist $a \in E$, $a' \in E'$ such that $\alpha = \bar{a}$, $\alpha' = \bar{a}'$ and $y = f(a, a')$. As an application, it is proved that if $A/\mathfrak{m}$ is a direct sum of two ideals $V/\mathfrak{m}$, $V'/\mathfrak{m}$, then A is a direct sum of the two ideals $\mathfrak{n} = \bigcap_n V^n$ and $\mathfrak{n}' = \bigcap_n V'^n$. *J. Dieudonné* (Evanston, Ill.).

Yoshii, Tensho. Note on algebras of strongly unbounded representation type. Proc. Japan Acad. **32** (1956), 383–387.

Let A be an associative algebra with a unity element over and algebraically closed field and $g_A(d)$ be the number of inequivalent indecomposable representations of A of degree d, where d is a positive integer. Then if A has indecomposable representations of arbitrarily high degrees and $g_A(d)$ is infinite for an infinite number of integers d the algebra A is said to be of strongly unbounded representation type. In his Michigan thesis J. P. Jans gave four conditions which are sufficient to imply that an algebra A be of strongly unbounded representation type, two of them being expressed in terms of the graph $G(A_0)$ associated with any two-sided ideal A_0 in the radical of A. The author derives two additional sufficiency conditions also expressed in terms of the graph $G(A_0)$.

A. A. Albert (New Haven, Conn.).

Kertész, A. Modules and semi-simple rings. II. Publ. Math. Debrecen **4** (1956), 229–236.

The author continues his study [same Publ. **3** (1954), 288–296; MR **17**, 342] of the reducibility of (left) R-modules and its connection with the semi-simplicity of rings with descending chain conditions. He obtains

conditions equivalent to the complete reducibility of a perfect R-module G (perfect $=RG=G$). He then characterizes semi-simple rings with descending chain conditions as follows: R is semi-simple if and only if for every left ideal L of R and for each g in any R-module G, the submodule Lg is a direct summand of G. This result implies a related characterization due to Goldman [Bull. Amer. Math. Soc. **52** (1946), 1021–1027; MR **8**, 433].

I. N. Herstein (New Haven, Conn.).

Bergmann, Artur. Über Ringe mit Normenadditionsformel. Math. Nachr. **15** (1956), 55–76.

Let $M=\{x, y, x_i\}$ denote an associative ring with unit on which an integer valued norm $[x]$ is defined for which the following axioms hold: (A_1) $[xy]=[x][y]$, (A_2) if $x\neq 0$ then $[x]\neq 0$, (A_3) there exists an integer n such that $(x_1, \cdots, x_{n+1})=0$ for any $x_i \in M$, but $(x_1, \cdots, x_k)=0$ for $k<n+1$, where inductively

$$(k+1)!(x_1, \cdots, x_k, x_{k+1})=k!(x_1, \cdots, x_{k-1}, x_k+x_{k+1})$$
$$-k!(x_1, \cdots, x_{k-1}, x_k)-k!(x_1, \cdots, x_{k-1}, x_{k+1}),\ (x)=[x].$$

Axiom (A_3) is a generalization of the addition formula of the norm introduced by H. Hasse in his proof of the Riemann hypothesis for elliptic fields. The author investigates the structure of the rings M subject to these axioms and the further axioms (A_4) $[x]\geqq 0$, and (A_5) M has an anti-isomorphism. Thus he obtains a generalization of the work of H. Behrbohm [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. Fachgruppe I. (N.F.) **1** (1935), 131–134] who treated the case $n=3$. First simple formal consequences of (A_1) to (A_3) are proved which in turn are used to establish that each $x \in M$ is the root of an equation

$$X(x)=\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}(x, \cdots, x, 1, \cdots, 1)x^{n-\nu}=0$$

with integral coefficients (the term x in the parentheses of length n occurs ν times, respectively). It follows easily that M is either an order of an algebraic number field of degree m, the maximal degree of the irreducible factors of the polynomials X, or it is an order in a division algebra of degree m_2 over its center which in turn has degree m_1 over the rational field, $m=m_1m_2$. Furthermore the author shows that more generally the norm of an algebraic extension of degree n over a field R (R is not necessarily the rational field, $[x] \in R$ this time) and the principal norm of a separable simple algebra A/R of rank $n_1n_2^2$ satisfy $(x_1, \cdots, x_{n+1})=0$, where $n=n_1n_2$ in the latter case. A combination of these results with some facts from the representation theory of commutative semi-simple algebras yields that the function $[x]$ must be a power of the norm or principal norm, respectively. Finally axioms (A_4) and (A_5) imply that the quotient algebra of M over the rational field is necessarily one of the division algebras or fields found in Albert's classification of generalized Riemann matrices. (A_4) in particular implies that n must be even. It should be noted that the multiplication rings of function fields of one variable do not necessarily always satisfy (A_2) [for examples see Lefschetz, Trans. Amer. Math. Soc. **22** (1921), 327–482, p. 463].

O. F. G. Schilling (Chicago, Ill.).

Patterson, E. M. Linear algebras of genus zero. J. London Math. Soc. **31** (1956), 326–331.

The genus of a linear algebra is the difference between its dimension (n) and the least number of linearly independent elements whose products generate the algebra. Algebras of genus zero are studied. The case where both $n=2$ and the coefficient field F is GF(2) is exceptional, it is discussed in detail. For all other cases a basis $\{e_i\}$ can be chosen such that the multiplication in the algebra is given by one of the following rules: (1) $e_ie_j=0$, (2) $e_ie_j=\lambda e_i+\mu e_j$ ($\lambda, \mu \in F$, not both zero), (3) $e_ie_j=\delta_j'e_i+\delta_i^2e_j$, where δ_i^j is the Kronecker symbol. *O. Taussky-Todd.*

See also: Weaver, p. 106; Samuel, p. 108; Paige, p. 110; Hodges, p. 113; Yen, p. 138.

Groups, Generalized Groups

★ **Speiser, Andreas. Die Theorie der Gruppen von endlicher Ordnung.** Birkhäuser Verlag, Basel und Stuttgart, 1956. xi+271 pp. 26 francs suisses.

Die vorliegende vierte Auflage wurde an einigen Stellen korrigiert (vgl. dritte Auflage, Springer, Berlin, 1927 und MR **7**, 410), ferner wurde in einem Anhang die Herstellung von Gruppenbildern erörtert und der Unterschied zwischen der funktionentheoretischen Auffassung und Substitution erörtert.

Aus dem Vorwort zur vierten Auflage.

★ **Weyl, Hermann. Symmetrie.** Birkhäuser Verlag, Basel und Stuttgart, 1955. 157 pp. DM 17.70.

A translation by Lulu Bechtolsheim of: Symmetry, Princeton University Press, Princeton, N.J., 1952 [MR **14**, 16].

Baer, Reinhold. Norm and hypernorm. Publ. Math. Debrecen **4** (1956), 347–350.

The norm $N(G)$ of a group G is the intersection of the normalisers of the subgroups of G. It is a characteristic subgroup of G containing the centre $Z(G)$. In general this inclusion is proper, but it is here shown that $Z(G)=1$ implies $N(G)=1$. It follows that if the hypernorm is defined as the intersection of normal subgroups X such that $N(G/X)=1$, then hypernorm and hypercentre coincide. *G. Higman* (Oxford).

★ **Berman, S. D. Group algebras of Abelian extensions of finite groups.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 7 pp.

A translation of the article reviewed in MR **17**, 235.

Hall, Marshall; and Paige, L. J. Complete mappings of finite groups. Pacific J. Math. **5** (1955), 541–549.

A complete mapping of a group G is a permutation S of G such that the mapping $T: xT=x\cdot xS$ is also a permutation of G. (If G has finite odd order, the identity mapping is complete.) Every infinite group has complete mappings [P. Bateman, Amer. Math. Monthly **57** (1950), 623–624; MR **12**, 670]. A necessary condition that a finite group G of even order have a complete mapping is that the Sylow 2-subgroups of G be non-cyclic [Bruck, Canad. J. Math. **3** (1951), 94–107; MR **12**, 580]. The present authors conjecture that this condition is also sufficient. They prove their conjecture for the symmetric, the alternating and the solvable groups and for other groups less easily described. *R. H. Bruck* (Madison, Wis.).

Ledermann, W.; and Neumann, B. H. On the order of the automorphism group of a finite group. II. Proc. Roy. Soc. London. Ser. A. **235** (1956), 235–246.

The authors study the divisibility of the order of the automorphism group of a finite group G by the powers of

the prime factors of the order of G. The technique, in many ways similar in spirit to that of part I [same Proc. 233 (1956), 494–506; MR 17, 580], is to consider the group G as a central extension of P, the Sylow p-subgroup of the center of G, by $F=G/P$. They first sharpen the bound on the order of the Schur multiplicator of F. From this they obtain a bound on the number of generators needed for Y, the subgroup of P generated by the factor set (of the representation of G as an extension of P by F). They show that the automorphisms of P which leave Y elementwise fixed can be extended to automorphisms of G. The discussion then turns to a discussion of two cases, viz. P has "many" generators (where "many" is defined) in which case there are many automorphisms of P which leave Y elementwise fixed, and to the case where P has "few" generators. In this case, as in their previous paper, they first imbed G in another group G^*. G^* is essentially the following: Let q=the order of the Sylow p-subgroup of F; let $v_1, \cdots, v_r$ be generators of P; introduce elements w_i so that $w_i^q=v_i$. Let P^* be the group generated by the w_i's; G^* is then defined by all formal products $x_\rho z^*$, $\rho \in F$, $z \in P$ (where the x_ρ are the coset representatives of P in G chosen in the representation of G as an extension) and define the multiplication of these formal products by the old factor sets $y_{\rho,\sigma}$ by $(x_\rho z_1^*)(x_\sigma z_2^*)=x_{\rho\sigma}y_{\rho,\sigma}z_1^* z_2^*$. Thus $G^*/P^* \approx G/P \approx F$. Every element of P now has a qth root in P^*. Letting $x_\rho^*=x_\rho z_\rho^*$ and using the corresponding factor set these generate a subgroup Y^* of P^*. The authors now, however, can explicitly define endomorphisms (in fact automorphisms) of P by: for a fixed index i define α by $w_i^\alpha=w_i^{1+k_\rho}$, $w_j^\alpha=w_j$ for $j\neq i$. This α leaves Y^* elementwise fixed so can be extended to an automorphism β of G^*. They then note that β maps G into itself. The problem then becomes one of determining when the restriction of β to G is the trivial automorphism. This they do and end up with the very interesting theorem: For every prime p there exists a function $g(h)$ such that if the order of a finite group G is divisible by $p^{g(h)}$ then the order of the automorphism group of G is divisible by p^h. The smallest such $g(h)$ satisfies: $g(1)=2$, $g(h)\leq(h-1)^2p^{h-1}+h$ for $h\geq 2$. *I. N. Herstein.*

Čunihin, S. A. Π-factorization of finite groups. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 397–399. (Russian)

Following nearly two pages of definitions, the author quotes his two theorems about finite groups, one concerning the existence of a subgroup with a certain order, the other concerning a certain type of factorization of the group into a product of subgroups. These results generalize and unify theories announced earlier [same Dokl. (N.S.) **95** (1954), 725–727; **97** (1954), 977–980; MR **16**, 10; **16**, 331]. *R. A. Good* (College Park, Md.).

Nagai, Osamu. On simple groups related to permutation-groups of prime degree. I. Osaka Math. J. **8** (1956), 107–117.

Let $\mathfrak{G}$ be a finite group which coincides with its commutator subgroup and which contains an element of prime order p that commutes only with its own powers. R. Brauer [Ann. of Math. (2) **44** (1943), 57–79; MR **4**, 266] has shown that the order of such a group has the form $p(p-1)(1+np)/t$, and determined all such groups for which $n<p+3$. Reynolds [Bull. Amer. Math. Soc. **61** (1955), 38–39] enumerated all such groups with $p+2<n\leq 2p-3$. The present paper extends the upper bound to $2p+3$. The methods used depend upon Brauer's theory of p-blocks of characters and the results are as follows: If $2p-3<n\leq 2p+3$, t is odd, $t>1$, then (1) $2p+1$ is a prime power $2p+1=l^a\geq 23$, where $l=3$ for $a>1$, and (2) $\mathfrak{G}\cong \mathrm{LF}(2, 2p+1)$, there being no group $\mathfrak{G}$ for which $2p-3<n<2p+3$. Further, if $n=2p+3$, $t=(p-1)/2$ then $2p+1$ is a prime power and $\mathfrak{G}\cong \mathrm{LF}(2, 2p+1)$ including LF(2, 7) and LF(2, 11). *H. K. Farahat* (Sheffield).

Gaschütz, Wolfgang. Erzeugendenzahl und Existenz von p-Faktorgruppen. Arch. Math. **7** (1956), 91–93.

If the group G is p-perfect (i.e. has no normal subgroup of index p), can be generated by e elements, and has a subgroup U of finite index with a normal subgroup $U_\mathfrak{p}$ such that $U/U_\mathfrak{p}$ is elementary abelian of order p^d, then

$$d\leq 1+(e-1)(G:U)-e,$$

and this inequality is best possible. *Graham Higman.*

Szép, J. Zur Theorie der Halbgruppen. Publ. Math. Debrecen **4** (1956), 344–346.

For each element a of a semi-group F, a set F_a is defined as follows: (i) If a generates an infinite cyclic group in F, and if $x^2=a$ implies that x commutes with the identity of this group, then F_a is this group. (ii) Otherwise, F_a is the semi-group generated by a.

The paper contains one theorem: There exists, for each a of a semi-group F, a subset C_a of F such that $F-F_a=F_aC_a$ and $F_a\cap C_a$ is null, if and only if there are disjoint sub-groups G_i of F (each with identity e_i) such that F is their union and $e_ie_j=e_j$. *H. A. Thurston.*

Pták, Vlastimil. Immersibility of semigroups. Časopis Pěst. Mat. **78** (1953), 259–261. (Czech)

A short account of the extract published by the author in Čehoslovack. Mat. Z. **2** (77), 247–271 (1952) [MR **15**, 598] from his article with the same title in Acta Fac. Nat. Univ. Carol. Prague no. **192** (1949) [MR **12**, 155].

Paige, Lowell J. A class of simple Moufang loops. Proc. Amer. Math. Soc. **7** (1956), 471–482.

The simple alternative rings which are neither associative nor nil are now completely known, the culminating theorem being that of Kleinfeld [Ann. of Math. (2) **58** (1953), 544–547; MR **15**, 392]. Such a ring is an eight-dimensional algebra over its centre F (which is a field) and is either (i) Zorn's "vector-matrix" algebra [Zorn, Abh. Math. Sem. Hamburg Univ. **8** (1931), 123–147; Albert, Canad. J. Math. **4** (1952), 129–135; MR **14**, 11] or (ii) a Cayley division algebra. The author studies certain multiplicative loops contained in these algebras (such loops are necessarily Moufang) and, in so doing, produces the first known examples of simple Moufang loops which are not groups.

In what follows, $Z(G)$ denotes the centre of the loop G.

A vector-matrix algebra R exists for every field F. In effect, the author introduces a "determinant" (the norm) for the elements of R, with the usual property $|xy|=|x|\,|y|$. He considers the "general linear" loop L (determinant not zero) and the "special projective" normal subloop M (determinant 1). $M/Z(M)$ turns out to be a simple Moufang loop which is finite or infinite with F and which is never associative. Two cases must be distinguished: (a) F of characteristic 2; (b) F of characteristic not 2. In case (a), $Z(M)=1$ and $L/Z(L)$ is isomorphic to $M/Z(M)$; if F has finite order 2^n, $M/Z(M)$ has order $2^{7n}-2^{3n}$. In case (b), $Z(M)$ has order 2 and $L/Z(L)$ contains a normal

subloop of index 2 which is isomorphic to $M/Z(M)$; if F has finite order p^n, $M/Z(M)$ has order $\frac{1}{2}(p^{7n}-p^{3n})$. {Reviewer's remark. These highly grouplike loops have some "anti-social" qualities; a letter from the author points out that his loop of order $120=2^3\cdot 3\cdot 5$ has no elements of order 5.}

In a concluding section the author considers the loop N of elements of norm 1 in the Cayley division algebra over the real field and proves that $N/Z(N)$ is a simple, not-associative Moufang loop. An example adapted from Dieudonné [Sur les groupes classiques, Hermann, Paris, 1948; MR 9, 494] shows that this result cannot be extended to arbitrary Cayley division algebras.

R. H. Bruck (Madison, Wis.).

Stein, Sherman K. Foundations of quasigroups. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 545–546.

A multiplicative system is a quasi-group if, when $ab=c$, any two of the three elements a, b, c uniquely determine the third. Any given quasi-group therefore automatically defines five others in which the operations are $b\cdot a=c$, $b\cdot c=a$, $c\cdot b=a$, $a\cdot c=b$, and $c\cdot a=b$. These six quasi-groups are called conjugates. Thus the six conjugates of a group G are the six quasi-groups obtained by defining the three operations ab (the group product), $a\circ b=a^{-1}b$ and $a*b=ab^{-1}$ and then forming the transposes ($a\times b=ba$ etc.) of these. In general only the transpose of the original group is associative. Several results concerning conjugate quasi-groups are announced and several conditions equivalent to the "medial law" $(ab)(cd)=(ac)(bd)$ are given. A quasi-group is defined whose elements are the points of a smooth convex plane curve K in such a way that the medial law is a necessary and sufficient condition that K be a conic. *D. C. Murdoch* (Vancouver, B.C.).

See also: Nöbauer, p. 111.

THEORY OF NUMBERS

General Theory of Numbers

Carlitz, L. A sum connected with quadratic residues. Nagoya Math. J. **10** (1956), 1–7.

If p is an odd prime, m an arbitrary positive integer, a an arbitrary integer, let

$$S_m(a)=\sum_{r=0}^{m}(-1)^{m-r}\left(\frac{r+a}{p}\right)\binom{m}{r},$$

where the third factor under $\sum$ is the Legendre symbol. The author considers the problem of finding $\theta_p(m,a)$, the highest power of p that divides $S_m(a)$. Using finite differences it is shown that $\theta_p(m,a)\geqq E_p(m)$, where $E_p(m)$ denotes the highest power of p that divides $m!$; and, using a formula for Gaussian sums, that $\theta_p(m,a)\geqq h$, where h is the integer defined by

$$(h-1)(p-1)<m-(p-1)/2\leqq h(p-1).$$

Moreover, $h\geqq E_p(m)$ and, indeed, the difference $h-E_p(m)$ may be arbitrarily large.

If l is the integer defined by $m+l=h(p-1)+(p-1)/2$, the author determines an explicit expression for the residue of $p^{-h}S_m(a)$ (mod p). In the special case $l=0$, $\theta_p(m,a)=h$, but this does not necessarily hold if $l\neq 0$. Other results are given for small values of l, for the case $p=3$, and some remarks are added concerning a sum analogous to $S_m(a)$, but without the factor $(-1)^{m-r}$.

R. D. James (Vancouver, B.C.).

Nöbauer, Wilfried. Eine Verallgemeinerung der eindimensionalen linearen Gruppe mod n. Monatsh. Math. **60** (1956), 249–256.

In earlier papers [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **162** (1953), 207–233; Monatsh. Math. **59** (1955), 194–202; MR **15**, 856; **17**, 131], the writer has studied the group $\mathfrak{G}_n$ consisting of elements of $\Gamma[x]/A$ with inverse; here Γ is the ring of rational integers, A is the ideal of residual polynomials (mod n) and the group operation is $f(g)$ where f, $g\in\Gamma[x]$. In particular it was shown that the study of $\mathfrak{G}_n$ can be reduced to the study of $\mathfrak{T}_{p'}$, where $\mathfrak{T}_{p'}$ denotes the subgroup of $\mathfrak{G}_{p'}$ with elements representable in the form

$$f(x)=a_0+a_1x+pa_2x^2+\cdots+p^{e-1}a_ex^e.$$

In the present paper it is proved that the set of elements of $\mathfrak{G}_n$ representable in the form

$$g(x)=a_0+a_1x+ta_2x^2+\cdots+t^{r-1}a_rx^r,$$

where the a_i are integers and t is a fixed positive integer, constitute a subgroup $\mathfrak{G}(n,t)$. The principal results of the paper are the following. I. If $n=ab$, where $(a,b)=1$, then $\mathfrak{G}(n,t)=\mathfrak{G}(a,t)\times\mathfrak{G}(b,t)$ (direct product). II. If $m|n$, then $\mathfrak{G}(m,t)$ is a homomorphic image of $\mathfrak{G}(n,t)$. III. If $s|t$, then $\mathfrak{G}(n,t)\subseteq\mathfrak{G}(n,s)$. IV. If $(p^e,t)=p^r$, then $\mathfrak{G}(p^e,t)=\mathfrak{G}(p^e,p^r)$. V. If $(n,s)=(n,t)$, then $\mathfrak{G}(n,s)=\mathfrak{G}(n,t)$; in particular if $s\equiv t$ (mod n) then $\mathfrak{G}(n,s)=\mathfrak{G}(n,t)$. VI. A system of representatives of $\mathfrak{G}(p^e,p^r)$ is given by

$$p(x)=a_0+\sum_{k=1}^{\infty}p^{r(k-1)}a_kx^k,$$

where

$$0\leqq a_0<p^e,\ 0\leqq a_1<p^e,\ a_1\not\equiv 0 \pmod p,$$
$$0\leqq a_k<p^{e-(k-1)r-\epsilon(k)}\ (k\geqq 2),$$

$\epsilon(k)$ denotes the exponent of the highest power of p dividing k. VII. Using the last result a formula iso btained for $o(e,r)$, the order of $\mathfrak{G}(p^e,p^r)$:

$$o(e,r)=(p-1)p^S,\ S=(e-1)=\sum_{k=1}^{u}(e-(k-1)r-\varepsilon(k)),$$

where u is the greatest solution of $(k-1)r+\varepsilon(r)<e$. A similar result is obtained for $o(e,0)$, the order of $\mathfrak{G}(p^e,1)$.

L. Carlitz (Durham, N.C.).

Cohen, Eckford. The number of solutions of certain cubic congruences. Pacific J. Math. **5** (1955), 877–886.

Formulas expressing the number of solutions of the congruences $n\equiv ax^3+by^3$ and $n\equiv ax^3+by^3+cz^3$ (mod p^λ) are given (p prime, a, b, c integers prime to p, n and λ integers). These formulas contain the integers A and B which are uniquely determined by the relations $4p=A^2+27B^2$, $A\equiv 1$ (mod 3), $B>0$. In a special case, however, the formula is explicit. The main tools used in the paper are Gauss sums.

H. Bergström (Göteborg).

See also: Weaver, p. 106; Rankin, p. 114; Bopp, p. 173.

Analytic Theory of Numbers

★ **Van der Corput, J. G. Sur le reste dans la démonstration élémentaire du théorème des nombres premiers.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 163–182. Georges Thone, Liège; Masson and Cie, Paris, 1956.

Starting from Selberg's celebrated inequality

$$(1)\quad \frac{\vartheta(x)}{x}+\frac{1}{x\log x}\sum_{p\leq x}\vartheta\Big(\frac{x}{p}\Big)\log p=2+O\Big(\frac{1}{\log x}\Big).$$

The author obtains

$$\vartheta(x)=x+O\Big(\frac{x}{(\log x)^{1/100}}\Big).$$

A little earlier P. Kuhn [Math. Scand. **3** (1955), 75–89; MR **17**, 587] proved that (1) implies $\vartheta(x)=x+O\Big(\frac{x}{(\log x)^{1/10}}\Big)$.

Starting from (1), (assuming for the error term on the right side only $o(1)$) he obtains a simplified version of his proof of the prime number theorem [Math. Centrum Amsterdam, Scriptum no. **1** (1948); MR **10**, 597]. *P. Erdős.*

★ **de la Vallée Poussin, Ch.-J. Sur la fonction $\xi(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 9–66. Georges Thone, Liège; Masson and Cie, Paris, 1956.

Reprinted from Mémoires de l'Académie royale de Belgique, Classe des Sciences, 1898, tome LIX.

★ **Errera, A. Sur le théorème fondamental des nombres premiers.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 111–118. Georges Thone, Liège; Masson and Cie, Paris, 1956.

La démonstration apporte une simplification à celle de Landau (Sitzungsberichte der preussischen Akademie der Wissenschaften, phys.-math. Klasse, 1932) et semble être, a son tour, devenue la démonstration analytique la plus courte, à ce jour. *From the introduction.*

Min, Szu-Hoa. A generalization of the Riemann ζ function. II. The order of $Z_{n,k}(s)$. Acta Math. Sinica **6** (1956), 1–11. (Chinese. English summary)

In this paper it is proved that for $A_1\leq\sigma\leq A_2$ we can find a constant A depending on A_1 and A_2 such that

$$Z_{n,k}(s)=O(t^A)$$

as $t\to\infty$. *Author's summary.*

★ **Ricci, Giovanni. Recherches sur l'allure de la suite $\left\{\frac{p_{n+1}-p_n}{\log p_n}\right\}$.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 93–106. Georges Thone, Liège; Masson and Cie, Paris, 1956.

Let $p_1<p_2<\cdots$ be the sequence of consecutive primes. Put $\varrho_n=(p_{n+1}-p_n)/\log p_n$. Denote by $D(x,a,b)$ the number of integers k satisfying $p_k<x$ and $a<\varrho_k\leq b$, $\omega(x,a,b)$ denotes

$$\frac{D(x,a,b)}{(b-a)x(\log x)^{-1}}.$$

The author calls u a point of condensation from the right of the sequence ϱ_n if there exist two numbers $\gamma>0$ and $h_0>0$ so that for each $0<h<h_0$ there exists a sequence $x_n\to\infty$ satisfying

$$(1)\quad D(x_n,u,u+h)>\gamma h\frac{x_n}{\log x_n}.$$

Left and two sided points of condensation are defined analogously, u is called point of condensation if

$$D(x_n,u-h,u+h)>\gamma h\frac{x_n}{\log x_n},$$

U denotes the set of points of condensation from the right, V from the left, and W denotes the points of condensation.

Clearly $U\subset W\subset X$ and $V\subset W\subset X$, where X is the set of all limit points of ϱ_n.

Westrynthuis proved that $\limsup\varrho_n=\infty$ (or $\infty\in X$) and the reviewer proved that X has positive measure and that it contains numbers less than 1.

The author proves among other results that $\inf U<1$, $\inf<V1$ and that U and V both have positive measure (in fact, he proves that both U and V have measure greater than $\frac{1}{8}$).

He further defines

$$^*\Lambda(a)=\liminf_{x=\infty}\frac{D(x,0,a)}{x(\log x)^{-1}},$$

$$\Lambda^*(a)=\limsup_{x=\infty}\frac{D(x,0,a)}{x(\log x)^{-1}}.$$

Clearly both $^*\Lambda(a)$ and $\Lambda^*(a)$ are non-decreasing functions satisfying $0\leq{}^*\Lambda(a)\leq1$, $0\leq\Lambda^*(a)\leq1$. A^* and Ω^* are defined by $\Lambda^*(A^*)=0$, $\Lambda^*(A^*+\varepsilon)>0$, $\Lambda^*(\Omega^*)=1$, $\Lambda^*(\Omega^*-\varepsilon)<1$ for every $\varepsilon>0$. $^*A$ and $^*\Omega$ are defined analogously. The author obtains inequalities for Ω^*-A^* and $^*\Omega-{}^*A$. Many problems are posed, e.g. is $^*A=A^*$ and $^*\Omega=\Omega^*$? Is it true that $^*A<\Omega^*$ and $^*\Omega\geq1$?

In general one would expect that $^*\Lambda(a)=\Lambda^*(a)$ for all a and $^*A=A^*=0$, $^*\Omega=\Omega^*=\infty$. *P. Erdős.*

Freĭman, G. A. Inverse problems of additive number theory. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 109–115. (Russian)

Let $\{a_r\}$ be a strictly increasing sequence of positive integers and denote by $q(N)$ the number of solutions of the inequality

$$(*)\quad a_1n_1+a_2n_2+\cdots\leq N$$

in non-negative integers $n_1, n_2, \cdots$. The normal procedure then consist in deriving information about $q(N)$ from given data relating to $\{a_r\}$. It is shown in the present paper that the converse procedure is equally feasible. The principal result proved is as follows. Let $A>0$, $0<\alpha<1$, and let β be an arbitrary real number. Then the asymptotic formula

$$\log q(N)\sim AN^\alpha(\log N)^{-\beta}\quad(N\to\infty)$$

holds if and only if

$$a_r\sim Dr^{(1-\alpha)/\alpha}(\log r)^{\beta/\alpha}\quad(r\to\infty),$$

where D can be expressed explicitly in terms of α and β. There is an analogous result for $q^*(N)$, the number of solutions of (*) with each n_i equal to 0 or 1. The main tools used in the proof are a Tauberian and an Abelian theorem of Hardy and Ramanujan [Proc. London Math. Soc. (2) **16** (1917), 112–132; Theorems A and B] and a theorem on Stieltjes integrals [Theorem 108 of Hardy, Divergent series, Oxford, 1949; MR **11**, 25]. The paper concludes with an application to a problem in statistical thermodynamics. *L. Mirsky* (Sheffield).

Hodges, John H. Weighted partitions for general matrices over a finite field. Duke Math. J. 23 (1956), 545–552.

Let $q=p^n$ and put

$$e(\alpha)=e^{2\pi i t(\alpha)/p},\ t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}} \text{ for } \alpha \in \mathrm{GF}(q).$$

If $A=(\alpha_{ij})$ is a square matrix with $\alpha_{ij} \in \mathrm{GF}(q)$, define $\sigma(A)=\sum \alpha_{ii}$, the trace of A. The paper is concerned with the weighted sum

$$S=S(B, W, R, A)=\sum_{UAV=B} e\{\sigma(UW+RV)\},$$

where B, W, R, A, U, V are matrices over $\mathrm{GF}(q)$, A non-singular of order m, B square of order t, $U'W$, R', V are $m\times t$ and the summation is over all U, V such that $UAV=B$. For $W=R=0$, the sum S reduces to $N_t(A, B)$, the number of solutions of $UAV=B$, which has been discussed in an earlier paper [same J. 22 (1955), 497–509; MR 17, 241]. In the general case

$$(*) \qquad S=q^{-t^2}(S_1+S_2+S_3),$$

where $S_1=q^{2mt}f(W, R)$ and $f(W, R)=1$ if $W=R=0$, and 0 otherwise; S_2 and S_3 are expressed in terms of certain generalized Kloosterman sums defined by

$$K_t(B, A)=\sum_{|\sigma|\neq 0} e\{-\sigma(BC+C^{-1}A)\}=K_t(A, B).$$

In particular when at least one of W, R is non-singular (*) reduces to

$$S=q^{t(m-t)}K_t(B, A_1),$$

where $A_1=RFEW$, $EAF=I$. A number of properties of the Kloosterman sums are obtained; in particular we mention

$$\sum_{U,V} K_t(B-UAV, W)=q^{mt}K_t(B, W),$$

where the summation is over all $U(t, m)$ and $V(m, t)$.

L. Carlitz (Durham, N.C.).

Tatuzawa, Tikao. Additive prime number theory in an algebraic number field. J. Math. Soc. Japan 7 (1955), 409–423.

The author calls an algebraic integer in an algebraic number field K a prime if the principal ideal which it generates is a prime ideal. Let J denote the set of all linear combinations of primes in K with rational integral coefficients and let I denote the set of all algebraic integers in K. It is proved that J is of finite index in I, where J and I are considered as additive groups, and that every number in J is expressible as the sum of a bounded number of primes (the bound depending on K). The proof is based on the work of Rademacher [Abh. Math. Sem. Hamburg. Univ. 3 (1924), 109–163, 331–378; Math. Z. 27 (1927), 321–426; S.-B. Preuss. Akad. Wiss. 1923, 211–218] and Siegel [Ann. of Math. (2) 46 (1945), 313–339; MR 7, 49] but uses Brun's method and Schnirelman's density theorem.

H. Davenport (London).

Shapiro, Harold N. Distribution functions of additive arithmetic functions. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 426–430.

Let $f(m)$ be an additive arithmetic function, set $A_x=\sum_{p<x} f(p)/p$, $B_x^2=\sum_{p<x} f^2(p)/p$ (p prime), and suppose that $Bq\to\infty$ as $x\to\infty$. The author proves that if, for any $\eta>0$,

$$(1) \qquad \sum_{p<n, |f(p)|>\eta B_n} f^2(p)/p=o(B_n^2)$$

as $n\to\infty$, the distribution of $(f(m)-A_n)/B_n$ ($m\leq n$) is asymptotically Gaussian; this sharpens a well-known theorem of Erdös and Kac [Amer. J. Math. 62 (1940), 738–742; MR 2, 42]. He conjectures that (1) is also a necessary condition for the truth of the Erdös-Kac theorem.

H. Halberstam (Exeter).

See also: Shapiro, p. 113; Kubilyus, p. 113; Deuring, p. 113; Čulanovskiĭ, p. 115; Richert, p. 123.

Theory of Algebraic Numbers

Kubilyus, I. The distribution of Gaussian primes in sectors and contours. Leningrad. Gos. Univ. Uč. Zap. 137. Ser. Mat. Nauk 19 (1950), 40–52. (Russian)

Hecke's results about the asymptotic distribution of primes in algebraic fields were sharpened by Rademacher [Math. Ann. 111 (1935), 209–228] in the case of real quadratic fields, in that he was able to obtain estimates for the error terms in the corresponding prime number theorems. In the present paper the author follows Rademacher's programme in the field $k(i)$, and his proof is based to a considerable extent on Rademacher's work. To obtain his results, Hecke used Weyl's criteria for the uniform distribution mod 1 of points in n dimensional space, and was therefore unable to estimate the error terms in his asymptotic relations; for this reason the author uses instead a method based on lemma 12, Chapter I, of Vinogradov's "Method of trigonometric sums in the theory of numbers" [Trudy Mat. Inst. Steklov. 23 (1947); MR 10, 599; 15, 941]. The author's principal result is: the number of Gaussian primes in the sector $0\leq\phi_1\leq\arg \mathfrak{p}\leq \phi_2\leq 2\pi$, $N(\mathfrak{p})\leq x$, is $(2/\pi)(\phi_2-\phi_1)\int_2^x du/\ln u+O(xe^{-c\sqrt{\ln x}})$.

H. Halberstam (Exeter).

Deuring, Max. Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins. III. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. 1956, 37–76.

[For parts I, II of this paper see same Nachr. 1953, 85–94; 1955, 13–42; MR 15, 779; 17, 17.] Let K be a singular elliptic function-field over the algebraic field k of constants. It is assumed that K has at least one prime divisor $\mathfrak{o}$ of degree 1. The multiplier ring R of K is an order in an imaginary quadratic field Σ, the elements of R may be identified with those of k. The ring classfield Ω_R of Σ is contained in k. The zeta function $\zeta(s, K)$ of K is defined by means of

$$\zeta(s, K)=\prod_{\mathfrak{p}} \zeta(s, K, \mathfrak{p}),$$

where the product is over all the prime divisors of k and $\zeta(s, K, \mathfrak{p})$ is the zeta function of $K/\mathfrak{p}$. Then

$$\zeta(s, K, \mathfrak{p})=\frac{(1-\chi(\mathfrak{p})N\mathfrak{p}^{1-s})(1-\overline{\chi}(\mathfrak{p})N\mathfrak{p}^{1-s})}{(1-N\mathfrak{p}^{-s})(1-N\mathfrak{p}^{1-s})},$$

where $|\chi(\mathfrak{p})|=1$ for regular $\mathfrak{p}$, $\chi(\mathfrak{p})=0$ for irregular $\mathfrak{p}$ (for definitions see review of II).

The principal result of the present paper is contained in the following theorem.

χ is a proper "Grössencharakter" of k; its conductor F_χ contains all the irregular places of K and no other prime divisors of k.

$$\zeta(s, K)=\zeta(s, k)\zeta(s-1, k)L(s-\tfrac{1}{2}, k, \chi)^{-1}L(s-\tfrac{1}{2}, k, \overline{\chi})^{-1},$$

where $\zeta(s, k)$ is the Dedekind zeta function of k and $L(s, k, \chi)$ is the Hecke L-function of k for the Grössencharakter χ. $\zeta(s, K)$ is a meromorphic function of s that

satisfies the functional equation

$$(NF_\chi)^{-s}\zeta(s, K)=(NF_\chi)^{-(2s)}\zeta(2-s, K).$$

L. Carlitz (Durham, N.C.).

Kawada, Yukiyosi. Class formations. III. J. Math. Soc. Japan **7** (1955), 453–490.

The extended Galois groups defined by A. Weil in his treatment of class field theory [same J. **3** (1951), 1–35; MR **13**, 439] are considered in the general framework of a class formation, in the sense of Artin and Tate [see Kawada, Duke Math. J. **22** (1955), 165–177; MR **16**, 907]. Also, the system of Weil groups is determined for the class formation arising from the maximal unramified extension of a function field in one variable over the field of the complex numbers, as established jointly by the author and Tate [Amer. J. Math. **77** (1955), 197–217; MR **16**, 799].

The general investigation of this paper is based on unpublished work of Artin and Tate. However, the exposition fails to provide an adequate access to the theory. At any rate, it is not feasible to describe the actual results here; the definition alone that is given for a system of Weil groups covers two and a half pages.

G. Hochschild (Princeton, N.J.).

Cohn, Harvey. A device for generating fields of even class number. Proc. Amer. Math. Soc. **7** (1956), 595–598.

Attention is drawn to two types of fields which have even class number, the first consisting of certain real quadratic fields and the second of certain totally real (abelian) cubic fields. The latter are generated by

$$x^3-mx^2-(m+3)x-1=0,$$

where

$$m=g-1+(k^2-2g-1)/\{g(g+1)\},$$

m, $g>0$, $k>1$ being integers and it being supposed that m^2+3m+9 is square-free. It is observed that comparison with Swinnerton-Dyer's table (unpublished) shows that almost all the early fields with even class number are in fact of this type. *H. Davenport* (London).

Cohen, Eckford. Binary congruences in algebraic number fields. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 120–122.

Let P denote a prime ideal of norm p^f in a finite extension of the rational field F, let α, β and ρ be integers in F and m and λ positive integers. With some weak restrictions upon α, β, ρ, λ and m, the number of solutions of the congruence $\alpha X^m+\beta Y^m+\rho\equiv 0 \pmod{P^\lambda}$ is expressed by a formula containing the generalized Jacobi sum. It is also shown that the congruence $\alpha X^m+\beta Y^m+\gamma Z^m+\rho\equiv 0 \pmod{P^\lambda}$ is solvable for all P of sufficiently large norm, subject to the condition $(\alpha\beta\gamma, P)=1$ and some restrictions on m. *H. Bergström* (Göteborg).

Rademacher, Hans. Zur Theorie der Dedekindschen Summen. Math. Z. **63** (1956), 445–463.

The author proves various results about the values of Dedekind sums

$$s(h, k)=\sum_{\mu \bmod k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),\quad (h, k)=1,\ k>0,$$

where $((x))=0$ for integers x, $=x-[x]-\frac{1}{2}$ otherwise. The following of his results should be mentioned here: (1) $s(h, k)<s(1, k)$ for $1<h<k$; (2) $s(h, k)>0$ for $0<h<\sqrt{(k-1)}$; (3) let Γ be the group of all modular matrices

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad ad-bc=1.$$

Then the function

$$\Psi(M)=b/d \text{ if } c=0,$$
$$=(a+d)/c-12\ \mathrm{sgn}\ c\cdot s(a, |c|)-3\ \mathrm{sgn}(c(a+d)) \quad \text{if } c\neq 0,$$

depends only on the class of conjugates of M. Further the values of Ψ are integers and

$$\Psi(M)=\Psi(-M),\ \Psi(M^{-1})=-\Psi(M).$$

Since two modular matrices are conjugate if the corresponding quadratic forms $cx^2+(d-a)xy-by^2$ are equivalent, it is possible to evaluate $s(a, c)$ if the class number h of quadratic forms of discriminant $\Delta=(a+d)^2-4$ is 1. In this way the author proves the interesting result: (4) If $c>0$ and $a(m-a)\equiv 1 \pmod{c}$, then

$$s(a, c)=\frac{(1-c)m}{12c}$$

for $m=0, \pm 1, \pm 3$. He remarks that similar results can be obtained for other small values of h and gives an example for $h=2$. *H. D. Kloosterman* (Leiden).

See also: Tatuzawa, p. 113.

Geometry of Numbers

Birch, B. J. A transference theorem of the geometry of numbers. J. London Math. Soc. **31** (1956), 248–251.

Let A be a convex body in n-space with volume V and distance function $F(x)$. Where I is the lattice of all points with integer coordinates, let Λ be the lower bound of the numbers t for which the bodies tA with centers on the points of I cover the whole space. If $\lambda_1, \cdots, \lambda_n$ be the successive minima of $F(x)$, defined by Minkowski, let $Q_k=2^n(\lambda_1\cdots\lambda_{k-1}\cdot\lambda_k^{n-k-1}V)^{-1}$. The principal result of the paper is that $\Lambda\leqq\frac{1}{2}(Q_k\lambda_k)$, $1\leqq k\leqq n$, if $Q_k\geqq n!$ Examples are given to show that both the above estimate as well as the estimate for Λ given by Kneser [Math. Z. **61** (1955); 429–434; MR **16**, 898] valid for $n=2$, $Q_2<2$, cannot be improved. Thus the best possible estimate for Λ is known for all 2-dimensional convex regions. *D. Derry.*

Rankin, R. A. On the minimal points of positive definite quadratic forms. Mathematika **3** (1956), 15–24.

The basic result proved is that if $\mathbf{y}_1, \cdots, \mathbf{y}_m$ are any m points on the surface of the unit sphere in n dimensional space, then

$$\sum \Delta^2(j_1, \cdots, j_n)\leqq(m/n)^n,$$

where the summation is over all choices of n distinct suffixes $j_1, \cdots, j_n$ from $1, \cdots, m$, and $\Delta(j_1, \cdots, j_n)$ denotes the determinant formed from the coordinates of $\mathbf{y}_{j_1} \cdots, \mathbf{y}_{j_n}$. From this are deduced results on the determinants of the sets of integer points at which a positive definite quadratic form in n variables attains some or all of its successive minima. In particular, if $n\geqq 4$ and $f(\mathbf{x})$ is a perfect form, there exist n integer points $\mathbf{x}_1, \cdots, \mathbf{x}_n$ at which f attains its minimum, such that

$$0<\{\det(\mathbf{x}_1, \cdots, \mathbf{x}_n)\}^2\leqq n^{-1}\gamma_n{}^n(1+1/n)^n,$$

where γ_n is Hermite's constant. Finally a result is ob-

tained concerning the packing (supposed symmetrical about the centre) of spherical caps of angular radius α on a sphere, which is an improvement on an earlier result [Proc. Glasgow Math. Assoc. **2** (1955), 139–144; MR **17**, 523] when α is just less than $\frac{1}{3}\pi$. *H. Davenport.*

Val'fiš, A. Z. On lattice points in multi-dimensional ellipsoids. XVII. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 3–64. (Russian)

As in previous papers of this series [see MR **4**, 132; **13**, 919; **17**, 133], $P_k(x)$ denotes the difference between the number of lattice points in the k-dimensional sphere $y_1^2+\cdots+y_k^2=x$ and its volume. Moreover, P_k and ϱ_k are defined by

$$\left.\begin{aligned} P_k&=\limsup_{n\to\infty}\\ \rho_k&=\liminf_{n\to\infty}\end{aligned}\right\}\ \frac{2P_k(n)}{D_k n^{\frac{1}{2}k-1}}$$

where $D_k=\pi^{\frac{1}{2}k}/\Gamma(\frac{1}{2}k)$.

The present paper gives improved estimates for the quantities P_k and ϱ_k. The results and methods are of such a detailed nature that they cannot be readily quoted in a review. *W. H. Simons* (Vancouver, B.C.).

Čulanovskiĭ, I. V. An elementary proof of the law of distribution of primes in the Gaussian field. Vestnik Leningrad. Univ. **11** (1956), no. 13, 43–62. (Russian)

For a decent domain D in the two-dimensional plane, the author proves that the number N of points (x, y) such that $x^2+y^2=p$ (where p is a prime; x and y integers), (x, y) in D, is given by

$$N=\frac{2L}{\pi\log R}+O\Big(\frac{R^2}{\log R(\log\log R)^k}\Big).$$

Here L is the number of lattice points contained in D, and R is the distance from the origin to the farthest point (or points) of D. Asymptotic formulae of this kind were first found by Hecke. [For related results, see also Ankeny, Amer. J. Math. **74** (1952), 913–919; MR **14**, 249; Haselgrove, J. London Math. Soc. **26** (1951), 273–277; MR **13**, 438; Kubilyus and Linnik, Trudy. Mat. Inst. Steklov. **38** (1951), 170–172; MR **15**, 249.] The elementary method used by the author is that of A. Selberg. *S. Chowla.*

See also: Pignedoli, p. 161; Kopzon, p. 163; Sretenskiĭ, p. 165; Chao and Weiner, p. 172.

ANALYSIS

Functions of Real Variables

★**Rado, T.; and Reichelderfer, P. V. Continuous transformations in analysis. With an introduction to algebraic topology.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXV. Springer-Verlag, Berlin-Göttingen-Heildelberg, 1955. vii+442 pp. DM 59.60.

This is a systematic treatise on the topological and measure-theoretical foundations of the theory of real functions of several real variables, particular emphasis being given to "a line of thought initiated by Banach, Geocze, Lebesgue, Tonelli and Vitali".

Part I, entitled "Background in Topology", is mainly concerned with the elements of algebraic cohomology theory. It begins with the elements of general topology and the essential facts about abelian groups. This section is intended by the authors also to serve as an introduction to algebraic topology. The treatment is based in a general way on the treatise of Eilenberg and Steenrod [Foundations of algebraic topology, Princeton, 1952; MR **14**, 398]. It is very detailed and careful. The cohomology theory presented here differs from that of Eilenberg-Steenrod in that the authors obtain directly the so-called reduced theory. This theory was introduced by Rado [Proc. Amer. Math. Soc. **4** (1953), 244–246; MR **14**, 784]. Rado's cohomology group $H^{p+1}(X, A)$ is group (not ring!) isomorphic to the $H^p(X, A)$ of Wallace-Spanier [see Spanier, Ann. of Math. (2) **49** (1948), 407–427; MR **9**, 523] if A is a closed subset of X.

Part II, Topological Study of Continuous Transformations in R^n, is based to a considerable extent on the work of Federer [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 611–616; MR **10**, 361]. The first topic is the orientation of frames. A frame is a triple $[X, Y, D]$, where X and Y are non-empty compact sets in R^n, $X\supset Y$, and $X-Y=D$ is non-empty, open, and connected. For such a frame $H^{n+1}(X, Y)$ is infinite cyclic. Given any two frames $[X, Y, D]$ and $[X', Y', D']$ there is a standard isomorphism of $H^{n+1}(X, Y)$ onto $H^{n+1}(X', Y')$. There is a canonical isomorphism (gotten from the exact sequence) of $H^n(S^{n-1})$ onto $H^{n+1}(E^n, S^{n-1})$, where S^{n-1} is the $(n-1)$-sphere and E^n is the n-cell in R^n. $[E^n, S^{n-1}, \text{int } E^n]$ is a frame. Using this with the canonical and standard isomorphisms one obtains a canonical isomorphism of $H^{n+1}(X, Y)$ onto $H^n(S^{n-1})$. One now fixes on one of the two generators of $H^n(S^{n-1})$. The image of this generator in $H^{n+1}(X, Y)$ is the preferred generator for $H^{n+1}(X, Y)$. Once this is chosen all the frames are oriented. The authors show how to obtain a "positive" orientation for frames by choosing successively particular generators for $H^1(S^0)$, $\cdots$, $H^{n+1}(S^n)$, $\cdots$.

The topological index is now defined in terms of frames. Let $T:\bar{D}\to R^n$ be continuous and suppose $x\notin$ frontier D. Consider any frame $[A, B, \Delta]$ such that $T(\bar{D})\subset A$, $T(\text{fr } D)\subset B$, and $x\in\Delta$. T induces a homomorphism $T_*:H^{n+1}(A, B)\to H^{n+1}(\bar{D}, \text{fr } D)$. The image of the preferred generator of $H^{n+1}(A, B)$ is then an integral multiple $\mu(x, T, D)$, independent of the particular choice of the frame $[A, B, \Delta]$. This multiple is the topological index. The standard properties of $\mu(x, T, D)$ are now obtained. The last part of Part II is devoted to the definition and discussion of multiplicity functions, and maximal model continua.

Part III, Background in Analysis, is an elegant short outline course in advanced real variable theory, not presupposing Parts I and II.

Parts IV and V are the heart of the book. These are entitled Bounded Variation and Absolute Continuity in R^n, and Differentiable Transformations in R^n. They carry out a program announced by the authors in Proc. Nat. Acad. Sci. **35** (1949), 678–681 [MR **11**, 588]. This program was the development of a theory of bounded variation, absolute continuity, and generalized Jacobians which should be "comparable in utility and scope with the classical one-dimensional theory". Very general results on the transformation of multiple integrals are presented.

The range of applicability of these formulae depends on the scope of essentially absolutely continuous mappings, which are discussed in great detail.

Part VI, Continuous Transformations in R^2, is on two levels: first a classical discussion of the topological index and related concepts in R^2, then an application and specialization of the results of IV and V to $m=2$. There is a discussion of h-averages, bounded variation and absolute continuity in the sense of Tonelli.

J. M. Danskin (Princeton, N.J.).

Reichelderfer, Paul V. On the product of absolutely continuous transformations in euclidean n-space. Rend. Circ. Mat. Palermo (2) **5** (1956), 5–42.

The main result of this paper constitutes an answer to an open question in the theory developed in the book by T. Radó and P. V. Reichelderfer, Continuous transformations in Analysis, Grundlehren der math. Wissenschaften, reviewed above. For terminology the reader is referred to this book which will be quoted as CT in the sequel. Let D', D'' be bounded domains in Euclidean n-space R^n, and let $T':D'\to D''$, $T'':D''\to R^n$ be bounded continuous mappings such that T' is essentially absolutely continuous (eAC) in D', and T'' is eAC in D'' (see CT, § IV. 4). The problem is to find necessary and sufficient conditions for the essential absolute continuity of the product mapping $T''T':D'\to R^n$. In the case $n=1$ this problem reduces to that of finding necessary and sufficient conditions for the absolute continuity of the composition $g[f(x)]$ of two absolutely continuous functions f, g of the single real variable x. According to a classical result of N. Bary [Math. Ann. **103** (1930), 185–248, 598–653], a necessary and sufficient condition in this case is the summability of the product $g'[f(x)]f'(x)$ on the set where it is defined. The main result of the present paper states that the exact analogue is valid in the n-dimensional situation considered above, provided that the derivatives occurring in the 1-dimensional theorem are replaced by the essential generalized Jacobians (see CT, IV. 4.3) of the mappings T', T''. Furthermore, if T', T'', $T''T'$ are essentially absolutely continuous, then the chain rule (familiar from classical analysis for ordinary Jacobians) holds almost everywhere in D' for the essential generalized Jacobians involved. The author notes that his results cover the various special cases previously treated in the literature. Certain modifications of some of the basic concepts in CT play an important role in the proofs. The concept of an essential maximal model continuum (e.m.m.c.) (see CT, II.3.3) is modified by replacing, in the definition of an e.m.m.c., the term indicator domain by the term model indicator domain, where "model" means that the domain is the inverse set under T of some domain in R^n. Furthermore, the concept of an maximal model continuum (m.m.c.) for (x, T, D) (see CT, II.3.1) is extended in a natural manner to the concept of an m.m.c. for (X, T, D), where X is a continuum in R^n. The concepts of essential sets (see CT, II.3.6) and of the topological index $\mu(x, T, D)$ (see CT, II.2.2) are extended and modified accordingly. *T. Radó* (Columbus, Ohio).

Pellicciaro, E. J. Critical sets of functions. Ann. Mat. Pura Appl. (4) **41** (1956), 289–300.

Let $f(x)$ be a single-valued real function of the n real variables $x_1, x_2, \cdots, x_n$ in a bounded connected open set R of an n-dimensional euclidean space S and also on the boundary R' of R. By a critical point of $f(x)$ in R the author means a point p such that

$$\{f(p+aE)-f(p)\}\{f(p+bE)-f(p)\}\geqq 0$$

for all unit vectors E and all scalars a, b for which $|a|>0$, $|b|<\varepsilon$, where ε is a suitable positive number. Following W. M. Whyburn [Bull. Amer. Math. Soc. **35** (1929), 701–708] the author defines a critical set $M(p)$ corresponding to a critical point p to be the union of all connected sets comprised of p and other critical points q for which $f(q)=f(p)$. These critical sets are studied after classification into four types: Type 0 having the limit point property with respect to sets of the other types; Types 1 and 2 being analogous to classical minimum and maximum points respectively; Type 3 consisting of all critical sets not of types 0, 1 or 2. *M. Marden.*

Dvoretzky, Aryeh. On a theorem of J. L. Walsh. Proc. Amer. Math. Soc. **7** (1956), 363–366.

In a recent paper [same Proc. **4** (1953), 69–75; MR **14**, 736] J. L. Walsh has proved certain theorems concerning the behaviour of the sequence $\{f_n^{(p)}(x)\}$ of the derivatives of order p of a convergent sequence $\{f_n(x)\}$. In the present paper the author assumes less than point-convergence of $\{f_n(x)\}$ and establishes a more general result. Unfortunately (as the author and G. R. MacLane have kindly pointed out to the reviewer) the theorem as stated is erroneous and the author's revised statement is: Let $f(x)$ and $f_n(x)$ $(n=1, 2, \cdots)$ be p times differentiable in the interval $a<x<b$, and let

$$\lim_{n=\infty}\inf_{x\in I, y\in I}|f_n(y)-f(x)|=0$$

for every open sub-interval I of (a, b). Then, given any $x_0\in(a, b)$ and any open sub-interval (α, β) of (a, b) containing x_0, the sequence $N=\{n\}$ can be written as the union of two (not necessarily both infinite) sequences $N_1=\{n_1\}$ and $N_2=\{n_2\}$ in such a way that for every n_1 there exists $x_{n_1}\in(\alpha, \beta)$ for which $f_{n_1}^{(p)}(x_{n_1})=f^{(p)}(x_0)$, while if N_2 is infinite, we have

$$\limsup_{n=\infty}\int_{x_0-h}^{x_0+h}|f_{n_2}^{(p)}(x)-f^{(p)}(x)|dx=o(h)$$

as $0<h\to 0$. Moreover, if x_0 is not a local extremum point (in the wide sense) of $f^{(p)}(x)$, then the sequence N_2 may be taken as finite and the x_{n_1} may be taken so that we have $\lim_{n_1=\infty} x_{n_1}=x_0$.

The proof as printed appears to be valid for this amended enunciation. The author remarks that there are analogous theorems for one-sided derivatives and for functions of several variables. *U. S. Haslam-Jones.*

Chen, Kien-Kwong. The coefficients of Denjoy-Fourier series. Acta Math. Sinica **4** (1954), 263–278. (Chinese. English summary)

Fedulov, V. S. Some questions on the representation of a function by singular integrals over a domain. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 97–107. (Russian)

Remarks about the representability of functions $f(x)$ integrable over a region G as limits of integrals

$$\int_G \Phi_r(u, x)f(u)\,du,$$

where $\Phi_r(u, x)$ is kernel majorizable in a suitable way, and x belongs to the Lebesgue set of the function f.

A. Zygmund (Chicago, Ill.).

Tarnawski, E. Continuous functions considered from the standpoint of Dini's conditions. Fund. Math. **43** (1956), 3–22.

Let D_ω be the class of all bounded continuous functions on the line which satisfy a generalized Dini condition of the form

$$(*) \qquad \int_0^1 \frac{f(x+t)-f(x)}{\omega(t)}\,dt \leqq M,$$

where M is independent of x, $\omega(t)$ is continuous, positive, and monotonic for $t>0$, $\omega(t)\to 0$ as $t\to 0$, and $\int_\tau^1 \omega(t)^{-1}dt\to\infty$ as $\tau\to 0$.

By D_ω^∞ is meant the class of all bounded continuous functions on the line for which the integral (*) is infinite at every x. The author establishes necessary and sufficient conditions which must be satisfied by ω_1 and ω_2 in order that $D_{\omega_1}\cap D_{\omega_2}^\infty$ be non-empty. Functions of the form $\sum a_n\phi(b_nx)$, where $a_n>0$, $\sum a_n<\infty$, $0<b_n<b_{n+1}$, $b_n\to\infty$, and ϕ is periodic and satisfies a Lipschitz condition, are studied in detail and provide examples. The results are compared to those of an earlier paper by the same author [Fund. Math. **42** (1955), 11–37; MR **17**, 245]. *W. Rudin.*

See also: Komatu, p. 155.

Measure, Integration

Hsu, L. C. On a kind of asymptotic integrals with integrands having absolute maximum at boundary points. Acta Math. Sinica **4** (1954), 305–316. (Chinese. English summary)

It is the purpose of this paper to investigate the asymptotic behavior of a class of integrals of the following form

$$J(N)=\int\cdots\int_D \phi(u_1,\cdots,u_n)[f(u_1,\cdots,u_n)]^N du_1\cdots du_n \quad (N\to\infty),$$

in which $f(u)=f(u_1,\cdots,u_n)>0$ is assumed to attain an effective absolute maximum at a certain boundary point $\xi=(\xi_1,\cdots,\xi_n)$ of D, D being a simply connected n dimensional closed domain in Euclidean n space.

From the author's summary.

★ **Carathéodory, C. Mass und Integral und ihre Algebraisierung. Herausgegeben von P. Finsler, A. Rosenthal, und R. Steuerwald.** Birkhäuser Verlag, Basel und Stuttgart, 1956. 337 pp. DM 38.50.

This book embodies a detailed account of the author's researches in the field of measure and integration in an abstract setting, initiated in S.-B. Math. Nat. Abt. Bayer. Akad. Wiss. **1938**, 27–69. Other papers of the author on this subject appeared in the same journal **1938**, 175–183; Ann. Scuola. Norm. Sup. Pisa (2) **8** (1939), 105–130, together with those reviewed in MR **1**, 304; **3**, 211; **4**, 269; **9**, 517. The first chapter gives the abstract basis, viz. a Boolean algebra (an element of which is called a "soma") based on the axioms of M. H. Stone, an addition $+$ (Verbindung) forming a commutative group (corresponding to the symmetric difference in sets) and a multiplication such that $AA=A$. Since the latter operation defines an order, it is possible to set up the notions of union and intersection. This leads to an exposition of the properties of somas which parallels those of the theory of sets. The second chapter is concerned with properties of classes of somas, additivity, multiplicity, conjunctivity (Verbindungstreue) and subtractivity: if with A and B the union, intersection, $A+B$, $A-AB$, respectively belong to the class. A ring of somas has all of these properties, of which some are then obviously redundant. A ring is complete if every union of a denumerable number of somas belongs to the ring. The question of the smallest complete ring containing a given class of somas is based on a brief discussion of transfinite numbers of the second class. Homomorphisms are mappings of one class of somas on another, such that the union of a denumerable number of somas is the union of the maps, and disjoint somas map into disjoint somas, so that a complete ring goes into a complete ring. The subclass $\mathfrak{N}$ of somas which maps into the null soma is then a complete ring which is total (vollkommen) in the sense that every subsoma of a soma of $\mathfrak{N}$ is also in $\mathfrak{N}$. Chapter three is devoted to the subject of point functions. Finite valued point functions can be defined in an obvious way and form a lattice. A general point function is a map of the real axis $-\infty<y<+\infty$ on subsomas of a soma A, which is monotonic non-decreasing in the sense that $y<z$ implies $S(y)\subseteqq S(z)$ and called a somenskala. Two somenskala S and T define the same point function if for $y<z$ it is true that $S(y)\subseteqq T(z)$ and $T(y)\subseteqq S(z)$. These define a unique pair of functions $\alpha(X)$, $\beta(X)$ on somas to real numbers monotone in X which are similar to the greatest lower bound and least upper bound of a point function on a set. These functions are not independent since $\beta(X)=\sup[\alpha(Y);\ Y\subseteqq X]$ and $\alpha(X)=\inf[\beta(Y);\ Y\subseteqq X]$. A monotone function β is the upper function of a point function if and only if the class $\mathfrak{B}(y)$ of somas such that $\beta(X)\leqq y$ contains a largest soma $S^o(y)$. The fourth chapter considers various kinds of operations on point functions treated largely through their upper and lower functions $\alpha(X)$ and $\beta(X)$. Questions of order, addition, multiplication, convergence of sequences including uniform convergence, and continuous functions of point functions are discussed, and also the transformation induced on a point function due to a homomorphism of somas. Chapter 5 discusses measure functions. In addition to the monotoneity of a function on somas to real numbers there is the usual definition of additivity, upper and lower semi-additivity, and complete additivity and their relations. There is also the notion of "union bounded" if $\varphi(\sum A_i)\leqq\sum\varphi(A_i)$ and "cover bounded" if $B\subseteqq\sum A_i$ implies $\varphi(B)\leqq\sum\varphi(A_i)$. The usual definition of measurability is based on the formula $\varphi(A)=\varphi(UA)+\varphi(A-UA)$ for all A for which the terms are significant. A soma function is called a "measure function" if (a) its domain is a complete ring, (b) it is coverbounded and (c) $\varphi(0)=0$. Then the set of φ-measurable somas form a complete ring on which φ is completely additive. Application is made to the Borel-Lebesgue measure. The sixth chapter is devoted to the definition of integral. As is to be expected a point function is said to be measurable relative to a measure function φ if one of its somenskala consists of somas in the measurability ring $\mathfrak{M}_\varphi$ of φ. The integral for a finite valued measurable point function relative to φ is defined in the obvious way and this leads easily to that for a bounded point function, since such is the uniform limit of a sequence of finite valued functions. For the general point function f measurable relative to φ, $\int_X f d\varphi$ is that measure function $\psi(X)$ absolutely continuous relative to φ, in the sense that $\varphi(X)=0$ implies $\psi(X)=0$, whose measurability ring includes that of φ, and for which $\alpha(A)\varphi(A)\leqq\psi(A)\leqq\beta(A)\varphi(A)$ for all subsomas A of the domain of f corresponding to α and β, for which $0<\varphi(A)<\infty$. If M is a

normal soma relative to φ, i.e. the union of a denumerable number of somas M_i for each of which $0<\varphi(M_i)<\infty$, then this is uniquely determined. The function $f\geqq 0$ is summable on M, if M is a normal soma for both φ and ψ. The Radon-Nikodym theorem evolves very simply, Jordan decompositions and Lebesgue decompositions of measure functions relative to each other are developed. The seventh chapter devoted to limit processes in integration brings a convergence theorem similar to the Lebesgue theorem in which the sequence of point functions is dominated by a positive function s for which $0<\int_M s\,d\varphi<\infty$. It also considers the completeness of what might be called spaces L^1 and L^2 together with a Riesz-Fischer theorem, and closes with an ergodic theorem [see MR 3, 211; 9, 517]. The balance of the book centers in further studies of measure functions. Chapter 8 is concerned with the construction of measure functions. A positive function $p(U)$ on a subset $\mathfrak{B}$ of a complete ring $\mathfrak{A}$, defines a measure function ψ on $\mathfrak{A}$ by the condition $\psi(A)=\inf[\sum_j p(U_j);\ U_j\in\mathfrak{B};\ \sum_j U_j\supseteq A]$ for an A denumerably covered by $\mathfrak{B}$, together with $\psi(A)=+\infty$, for any A not covered by $\mathfrak{B}$. If $p(U)$ is the measure function $\varphi(U)$ and $\psi(X)=\varphi(X)$ for all X, then $\mathfrak{B}$ is a basis for φ. Chapter 9 treats regular measure functions also called outer measures which are measure functions μ^* which have at least one basis consisting of somas measurable for μ^*. Inner measure μ_* depends on the observation that the function $\mu^*U-\mu^*(U-X)$ is independent of U for μ^*-measurable somas U containing X for which $\mu^*U<\infty$. This gives rise to the consideration of properties of μ_* and the usual criterion for measurability involving μ^* and μ_*. The average: $\frac{1}{2}(\mu^*X+\mu_*X)$ is again a measure function whose measure ring contains that of μ^* and is absolutely continuous relative to μ^* but not necessarily regular. Chapter 10 treats regular measure functions which are similar („gleichartig") in the sense that they have the same domain of definition $\mathfrak{A}$ and basis $\mathfrak{B}$. Questions of convergence, decomposition, difference of such functions are considered. In particular if λ^* is a regular measure function and $f\geqq 0$ is summable relative to λ^*, then the integral $\mu^*X=\int_X f\,d\lambda^*$ is a similar regular measure. The eleventh chapter entitled: Inhaltsfunktionen or content function generalizes the functions defined by H. Hahn. If μ^*X is an outer measure on a complete ring $\mathfrak{A}$, and if for every normal soma A of $\mathfrak{A}$, there exists a sequence $U_1, \cdots, U_n, \cdots$, in the measurability ring of φ, which form a base for every subsoma of A, then μ^* is a content function. By setting up a canonical system of U's and subjecting A and μ^* to the homomorphism which maps all somas of null measure on the null soma, it is possible to map the reduced sets and measure function isomorphically and measure preserving on a complete ring of Lebesgue measurable sets on the straight line. This reduces Lebesgue integration as defined e.g. in ∞-dimensional space by B. Jessen to integration on the straight line. The chapter concludes with a Vitali covering theorem in n-dimensional space, the relationship between Lebesgue integration and derivatives in n-dimensions, and a linear measure which reduces to the length of a curve if the curve has a length, but provides also a measure for disconnected sets. An appendix shows that the axiomatic basis of the original paper on somas is equivalent to that used in this book. It concludes with an example of a class of somas which are not isomorphic in the sense of the author to the subsets of a set. With the exception of an incidental reference to J. Ridder [Acta Math. 73 (1941), 131–173; MR 3, 206; see also Nederl. Akad. Wetensch., Proc. 49 (1946), 167–174; MR 7, 513] no mention is made of work along similar lines [e.g. J. M. Olmsted, Trans. Amer. Math. Soc. 51 (1942), 164–193; MR 4, 11, A. P. Gomes, Portugal. Math. 5 (1946), 1–120, 218; MR 8, 275, 276].

T. H. Hildebrandt (Ann Arbor, Mich.).

Pagni, Mauro. Sulla derivazione negli insiemi astratti delle funzioni a variazione limitata integrabili secondo Burkill. Rend. Sem. Mat. Univ. Padova **25** (1956), 279–302.

The setting is taken from G. Fichera's Lezoini sulle trasformazioni lineari, vol. I [Istituto Matematico, Università, Trieste, 1954; MR 16, 715]. $(I_0, \mathfrak{B}, \mu)$ is a measure space, $\mu(I_0)<\infty$, $\{I\}$ a semiring $\subset\mathfrak{B}$, $I_0\in\mathfrak{B}$. For a real (finite) non-negative function F defined on $\{I\}$, $\theta[F, I_0]$ denotes the set of the real non-negative μ-summable functions f on I_0 such that $J_I(f)=\int_I f(x)d\mu\leq F(I)$ for every I. Any function f^* in $\theta[F, I_0]$ making $J_{I_0}(f)$ to a maximum is called a μ-derivative of F in Fichera's sense. f^*, if it exists, is defined except for a μ-nullset and denoted by F'. Theorem: If F is super-additive, F' exists and is μ-equivalent to the μ-derivative of the Burkill integral of F. {Remarks by the reviewer: 1) Definitions and results of the paper concerning the abstract Burkill theory not mentioned above, can be substantially found in a more general form in Haupt, Aumann and Pauc, Differential- und Integralrechnung, vol. 3 [2nd ed., de Gruyter, Berlin, 1955, §§ 10.1–10.2; MR 17, 1066]. 2) The author's "subadditività" is the usual "super-additivity"}.

C. Y. Pauc (Lafayette, Ind.).

Krickeberg, Klaus. Extreme Derivierte von Zellenfunktionen in Booleschen σ-Algebren und ihre Integration. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 217–279 (1956).

In C. R. Acad. Sci. Paris **236** (1953), 1937–1939 [MR 15, 205] the reviewer started a Burkill theory of interval functions for abstract intervals called "cells". In order to save the usual expensive Vitali conditions [Haupt, Aumann and Pauc, Differential- und Integralrechnung, Bd. 2. Aufl., de Gruyter, Berlin, 1955, § 10.1–10.2; MR 17, 1066] and thus give to the theory a larger field of applications, the classical pointwise derivatives were replaced by Radon-Nikodym integrands and global derivatives. In the present memoir the author, resorting to the complete lattice property of measurable functions with regard to essential ordering, introduces "essential derivates" and shows the necessity of Vitali conditions te secure a Burkill inequality (Th. 2.4) for such derivates. The use of the essential derivates saves the cumbersome restrictions (e.g. condition **L**) imposed by the use of pointwise derivates when their measurability is needed [Haupt and Pauc, Portugal. Math. 13 (1954), 37–54; MR 16, 805]. Some results were announced by the author in a previous note [C.R. Acad. Sci. Paris 238 (1954), 764–766; MR 15, 610].

Setting in § 1–3. $\mu|\mathcal{B}$: positive σ-finite measure defined on the abstract Boolean σ-algebra $\mathcal{B}$. E: unit of $\mathcal{B}$. A $\mathcal{B}$-function on E is defined by means of a spectral scale (spectral decomposition of E). The family $\mathcal{F}$ of the $\mathcal{B}$-functions is a complete lattice; sup and inf are denoted by $\vee$ and $\wedge$. For $f, g\in\mathcal{F}$, $f=g\,[A]$ means that the restrictions of f and g to A are equal. $f\in\mathcal{F}$ is integrable if the integrals over E of the positive and negative parts are not both infinite, summable if they are both finite. Θ or $(\Theta, \lessdot)$: set of elements $\sigma, \tau, \cdots$ directed by $\lessdot$. $\mathfrak{S}=(f_\sigma)$: Moore-Smith sequence in $\mathcal{F}$. $g_\tau=\bigvee_{\sigma\lessdot\tau} f_\sigma$, $h_\tau=$

$\wedge_\tau \ll_\sigma f_\sigma$, $\limsup f_\sigma = \wedge g_\tau$, $\liminf = f_\sigma = \vee h_\tau$. § 1 deals with the commutativity of integration and passage to the limit for $\mathfrak{S}$. A Moore-Smith sequence (φ_σ) of real σ-additive functions defined on $\mathcal{B}$ is called "terminally uniformly absolutely continuous from above" if to any $\varepsilon>0$ there exists a terminal set Δ in Θ, a $\delta>0$ and an element H in $\mathcal{B}$ of finite measure such that

$$((\sigma \in \Delta)\&(A \in \mathcal{B})\&(\mu(H.A)<\delta)) \rightarrow (\varphi_\sigma(A)<\varepsilon).$$

Theorem 1.3 (Generalized Fatou-Vitali Lemma): If $\limsup f_\sigma$ is integrable, if terminally many f_σ are integrable and if the sequence of their indefinite integrals is terminally uniformly absolutely continuous from above, then $\limsup \int_E f_\sigma \cdot d\mu \leq \int_E (\limsup f_\sigma) \cdot d\mu$. Condition **F2** for $\mathfrak{S}$: All f_σ are integrable; to any $\varepsilon>0$ and any $\varrho \in \Theta$ there exists an index ζ with $\varrho \ll \zeta$ such that for any finite subset $\{\sigma_1, \cdots, \sigma_p\}$ of Θ with $\zeta \ll \sigma_k$, $k=1, \cdots, p$, there is an index ξ with $\zeta \ll \xi$ and $\int_E \vee_k f_{\sigma_k} \cdot d\mu \leq \int_E f_\xi \cdot d\mu + \varepsilon$. Theorem 1.5: If there exists $\theta \in \Theta$ such that g_τ is integrable $\int_E g_\tau \cdot d\mu < +\infty$ and $-\infty < \limsup \int_E f_\sigma \cdot d\mu$, then the assertions

$$\int_E (\limsup f_\sigma) \cdot d\mu \leq \limsup \int_E f_\sigma \cdot d\mu,$$

$$\int_E (\limsup f_\sigma) \cdot d\mu = \limsup \int_E f_\sigma \cdot d\mu,$$

and **F2** are equivalent. § 2 deals with integration and differentiation of cell functions. $\mathcal{K}$: non-empty set of elements of $\mathcal{B}$ of finite, positive measure called "cells". Cell partition of E: (enumerable) disjoint subset $\mathcal{S}$ of $\mathcal{K}$ with $\vee\mathcal{S}=E$. A set $\mathcal{A} \subseteq \mathcal{B}$ is $\mathcal{S}$-fine if every $\mathcal{A}$-set is part of an $\mathcal{S}$-cell. $\mathfrak{T}$ or $(\mathfrak{T}, \ll)$: directed set of cell partitions of E (special instance of Θ in § 1). For $\mathcal{S}, \mathcal{T} \in \mathfrak{T}$, $\mathcal{T} \sqsubset \mathcal{S}$: $\mathcal{S}$ is $\mathcal{T}$-fine. ψ (cell function) : real function defined on $\mathcal{K}$. $\psi(\mathcal{S}) : \Sigma\psi(S)$ for $S \in \mathcal{S}$ if it exists. Provided $\psi(\mathcal{S})$ exists for a terminal subset of $\mathfrak{T}$,

$$\underline{\int}_E \psi = \liminf \psi(\mathcal{S}), \quad \overline{\int}_E \psi = \limsup \psi(\mathcal{S}).$$

$D(\psi, \mathcal{S})$ ($\mathcal{S}$-derivation of ψ): $\mathcal{B}$-function taking on each $S \in \mathcal{S}$ the value $\psi(S)/\mu(S)$. $\bar{D}\psi$ (upper derivate) $=\limsup D(\psi, \mathcal{S})$. $\underline{D}\psi = \liminf D(\psi, \mathcal{S})$. A set $\mathcal{L} \subset \mathcal{K}$ is called a "fine covering" of $A \in \mathcal{B}$ if for any terminal subset $\mathfrak{D}$ of $\mathfrak{T}$, $A \leq \vee(\mathcal{L} \cdot \mathbf{U}\mathfrak{D})$. Strong Vitali condition **V**(for $\mathfrak{T}$): Corresponding to any $A \in \mathcal{B}$, any fine covering $\mathcal{L}$ of A any terminal subset $\mathfrak{G}$ of $\mathfrak{T}$ and any $\varepsilon>0$, there exists a finite disjoint subset $\mathcal{P}$ of $\mathcal{L}$ included in some partition of $\mathfrak{G}$ such that $\mu(A - A \cdot \vee\mathcal{P}) < \varepsilon$. Theorem 2.4: If $\mu(E)<\infty$ and for any non-negative μ-Lipschitzian cell function ψ, $\int_E \bar{D}\psi \cdot d\mu \leq \overline{\int}_E \psi$, then $\mathfrak{T}$ fulfills **V**. The restriction $\mu(E)<\infty$ can be omitted if $\ll = \sqsubset$. Hint to the proof: Auxiliary function $\psi(K)=\mu(K)$ for $K \in \mathcal{L}$, $=0$ for $K \in \mathcal{K} - \mathcal{L}$. Let $\mathcal{Z} \in \mathfrak{T}$, $\mathfrak{D}$ be the set of the $\mathfrak{T}$-partitions $\geq \mathcal{Z}$ and $\bar{D}_{\mathcal{Z}}\psi = \vee D(\psi, \mathcal{S})$ for $\mathcal{S} \in \mathfrak{D}$. By hypothesis $A \leq \vee(\mathcal{L} \cdot \mathbf{U}\mathfrak{D})$, hence $1 = \bar{D}_{\mathcal{Z}}\psi[A]$. In virtue of Th. 1.5, $0 \leq \psi$, $0 \leq \bar{D}_{\mathcal{Z}}\psi \leq 1$ and $\mu(E)<\infty$, there exists $\mathcal{Z}$ and $\mathcal{X}$ in $\mathfrak{G}$ with $\mathcal{Z} \ll \mathcal{X}$ $\int_E (\bar{D}_{\mathcal{Z}}\psi - D(\psi, \mathcal{X})) \cdot d\mu < \varepsilon$. Take $\mathcal{P} = \mathcal{X} \cdot \mathcal{L}$. Definition: The cell function ψ is called $\mathfrak{T}$-absolutely continuous if to any $\varepsilon>0$, there exists $\delta>0$, a terminal set $\mathfrak{D}$ in $\mathfrak{T}$ and H in $\mathcal{B}$ of finite measure such that for any subset $\mathcal{S}$ of a partition of $\mathfrak{D}$ with $\mu(H \cdot \vee\mathcal{S}) < \delta$, $\psi(\mathcal{S})$ exists and $|\psi(\mathcal{S})| < \varepsilon$. Theorem 2.7: If for $\ll = \sqsubset$, the cell function ψ is $\mathfrak{T}$-absolutely continuous and of bounded $\mathfrak{T}$-variation, then $\int_E \psi = \int_E \underline{D}\psi \cdot d\mu$, $\overline{\int}_E \psi = \int_E \bar{D}\psi \cdot d\mu$, and the integrals are finite. Several results of § 2 (e.g. Th. 2.7) are formulated for $\ll = \sqsubset$. In § 3 the author exhibits conditions on $(\mathfrak{T}, \ll)$ securing the coincidence of the $\ll$-derivates and $\ll$-integrals with the $\sqsubset$-derivates and $\sqsubset$-integrals for universally absolutely continuous cell functions. Condition **R** expresses substantially the validity of a lemma of D. Rutovitz (O. Haupt-G. Aumann-Chr. Pauc, loc. cit., p. 232). Condition **D**: For any $\mathcal{T} \in \mathfrak{T}$ and any disjoint finite set $\mathcal{J} \subseteq \mathcal{K}$ such that corresponding to any cell $J \in \mathcal{J}$ there is $\mathcal{S} \in \mathfrak{T}$ with $J \in \mathcal{S}$ and $\mathcal{S} \gg \mathcal{T}$, there exists $\mathcal{Z}$ in $\mathfrak{T}$ with $\mathcal{T} \ll \mathcal{Z}$ and $\mathcal{J} \subseteq \mathcal{Z}$. Under condition **D**, **V** is equivalent with Condition $\mathbf{V}_0$: For any $\varepsilon>0$, any fine covering of any $A \in \mathcal{B}$ of finite measure includes a finite disjoint set $\mathcal{P}$ with $\mu(A - A \vee \mathcal{P}) \leq \varepsilon$.

In § 4 and 5, $(E, \mathcal{B}, \mu)$ is a measure space, μ is σ-finite, μ^* the completion of μ, $\mathfrak{N}$ the set of the μ-nullsets, $\mathfrak{N}^*$ the σ-ideal of the μ^*-nullsets. The definitions of § 1–3 are transposed via $\mathcal{B}/\mathcal{N}$ and the quotient measure. Thus $\mathcal{L} \subseteq \mathcal{K}$ is a fine covering of a set $A \in \mathcal{B}$ if, for any terminal subset $\mathfrak{D}$ of $\mathfrak{T}$, $\mathcal{L} \cdot \mathbf{U}\mathfrak{D}$ includes an enumerable subset $\mathcal{M}$ with $A \subset \mathbf{U}\mathcal{M}$ mod $\mathcal{N}$. Condition **V** becomes: To any $\varepsilon>0$, any terminal subset $\mathfrak{G}$ of $\mathfrak{T}$ and any fine covering $\mathcal{L}$ of a set $A \in \mathcal{B}$ of finite measure, there exists a finite subset $\mathcal{P}$ of $\mathcal{L}$ included in a partition of $\mathfrak{G}$ for which $\mu(A - A \cdot \mathbf{U}\mathcal{P}) < \varepsilon$. The pointwise derivates of a cell function ψ are denoted by $\bar{D}^*\psi$ and $\underline{D}_*\psi$. Condition **L**: Any fine covering $\mathcal{L}$ mod $\mathcal{N}^*$ of any set $M \subseteq E$ includes an enumerable cofinal subset covering M mod $\mathcal{N}^*$. Theorem 4.2: Let $\mathfrak{T}$ admit an enumerable cofinal subset. Then $\bar{D}\psi \leq \bar{D}^*\psi$ mod $\mathcal{N}^*$ holds for any cell function. Condition **L** is necessary for $\bar{D}\psi = \bar{D}^*\psi$ mod $\mathcal{N}^*$, ψ denoting any non negative μ-Lipschitzian cell function, and suffices to secure the validity of this equality for any cell function. § 5 gives examples. $\mathcal{K}$=set of all $\mathcal{B}$-sets of finite positive measure. $\mathfrak{T}$=set of all partitions of E. $\ll = \sqsubset$. De la Vallée Poussin-Jessen nets ($\ll = \sqsubset$) satisfy $\mathbf{V}_0$ but not **V**.

Chr. Pauc (Lafayette, Ind.).

Ogasawara, Tôzirô; and Yoshinaga, Kyôichi. Extension of $\natural$-application to unbounded operators. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 273–299.

The authors continue their non-commutative integration theory [same J. **18** (1955), 311–347; MR **17**, 66], extending the $\natural$-mapping to unbounded operators of a ring. [For similar extensions, see Dixmier, Compositio Math. **10** (1952), 1–55; MR **14**, 481.] A linear system $\mathfrak{S}$ of operators whose positive parts have a $\natural$ map is investigated, and found to have the following properties: 1) $\mathfrak{S}$ is an algebra if and only if the ring is of type I. 2) $\mathfrak{S}^2 \supseteq \mathfrak{S}$ if and only if the ring is finite.

The preceding results are used to define Lebesgue spaces of operators and to prove a monotone convergence theorem and the Radon-Nikodym theorem of Segal [Ann. of Math. (2) **57** (1953), 401–457; **58** (1953), 595–596; MR **14**, 991; **15**, 204]. *E. L. Griffin.*

Rechard, Ottis W. Invariant measures for many-one transformations. Duke Math. J. **23** (1956), 477–488.

Es sei $\mathfrak{X}$ eine sigma-Algebra von Teilmengen einer Menge X mit $X \in \mathfrak{X}$, m ein endliches Maß auf $\mathfrak{X}$ und τ eine eindeutige, meßbare und nichtsinguläre Transformation von X in sich. Ein zweites derartiges Maß m^* heißt stärker als m, wenn aus $m^*(A)=0$ folgt $m(A)=0$, und τ-asymptotisch stärker als m, wenn aus $m^*(A)=0$ folgt

$$\inf_n m(\tau^{-n}(A)) = 0.$$

Beide Begriffe fallen zusammen, wenn τ umkehrbar eindeutig mit meßbarer Umkehrung ist. Einen sich nur auf diesen Fall beziehenden Satz von Cotlar und Ricabarra

[Rev. Un. Mat. Argentina **14** (1949), 49–63; MR **12**, 85] verallgemeinernd beweist der Verf., daß dann und nur dann ein gegenüber τ invariantes Maß m^* existiert, so daß m stärker als m^* und m^* τ-asymptotisch stärker als m ist, wenn die in $\mathfrak{X}$ erklärten Mengenfunktionen $m(\tau^{-n}(E))$, $n=1, 2, \cdots$ gleichmäßig totalstetig in bezug auf m sind. Dies dient dazu, die Beweise der Ergodensätze von Dunford und Miller [Trans. Amer. Math. Soc. **60** (1946), 538–549; MR **8**, 280], denen ein nicht invariantes Maß zugrunde liegt, zu vereinfachen. Das Hauptergebnis kann teilweise auf den Fall einer Abelschen Semigruppe meßbarer Transformationen übertragen werden.

K. Krickeberg (Madison, Wis.).

See also: Freiman, p. 112; Rado and Reichelderfer, p. 115; Masani, p. 138.

Functions of Complex Variables

Arrighi, Gino. Sur la formule de Buchheim. C. R. Acad. Sci. Paris **242** (1956), 3030–3032.

Given a many-valued function $\varphi(z)$ and a square matrix A of order n, a polynomial $P(z)$ is constructed having the following property: if ϱ is a latent root of A of multiplicity m, then

$$\left[\left(\frac{d}{dz}\right)^j P(z)\right]_{z=\varrho} = \left[\left(\frac{d}{dz}\right)^j \phi(z)\right]_{z=\varrho} \quad (j=1, 2, \cdots, m),$$

where different branches of $\varphi(z)$ may be chosen for different latent roots ϱ provided only that the branch is regular at $z=\varrho$. An explicit expression for $P(z)$ is obtained by means of Lagrange's interpolation formula. The author has elsewhere discussed the problem in greater detail, with full historical references [Ann. Scuola Norm. Sup. Pisa (3) **8** (1954), 141–156; MR **17**, 252].

W. Ledermann (Manchester).

Choquet, Gustave. Fonctions analytiques et surfaces de Riemann. Enseignement Math. (2) **2** (1956), 1–11.

This short article gives an introductory account of analytic continuation and of an analytic function in the sense of Weierstrass based on currently used topological notions. *M. Heins* (Princeton, N.J.).

Serrin, James. A note on harmonic functions defined in a half plane. Duke Math. J. **23** (1956), 523–526.

New proofs are given for the Julia-Wolff theorem and the sharp form of the Phragmén-Lindelöf principle for harmonic functions in a half-plane. The relation between the Julia-Wolff theorem and the thus restricted Phragmén-Lindelöf theorem is pointed out. *M. Heins.*

Nehari, Zeev; and Singh, Vikramaditya. On the conformal mapping of nearly-circular domains. Proc. Amer. Math. Soc. **7** (1956), 370–378.

Let D be a nearly circular domain bounded by a smooth Jordan curve C with the equation $r=1+\varepsilon p(\theta)$ in polar coordinates, where ε is a small positive number. An approximation formula accompanied by a remainder term $O(\varepsilon^2)$ is derived for the Szegö kernel function of D. The error in approximation is estimated in terms of the quantities expressing the deviation of C from the unit circumference. *Y. Komatu* (Tokyo).

Buchner, P. Eine Anwendung der Theorie der komplexen Zahlen. Elem. Math. **11** (1956), 73–75.

An example suitable for treatment in school. With the help of elementary computation and simple geometrical construction two figures are drawn. The first gives the mapping by $f(z)=z+z^{-1}$ of the circle with center $z=i$ and radius $R=\sqrt{2}$. In the second a Joukowski profile is obtained from a slightly different circle. *B. Amirà.*

Kuramochi, Zenjiro. Mass distributions on the ideal boundaries of abstract Riemann surfaces. I. Osaka Math. J. **8** (1956), 119–137.

A detailed account is given of the work reported by the author in Proc. Japan Acad. **32** (1956), 1–6 [MR **17**, 1072].

M. Heins (Princeton, N.J.).

Huckemann, Friedrich. Zur Darstellung von Riemannschen Flächen durch Streckenkomplexe. Math. Z. **65** (1956), 215–239.

The object of the present paper is to extend the notion of a "Streckenkomplex" in such a manner that the generalized "Streckenkomplex" can be used to represent concrete simply-connected Riemann surfaces whose ramification points and boundary elements lie over a closed Jordan curve, finiteness conditions being dropped. In the hyperbolic case it is assumed that there are no free Jordan arcs. The paper proceeds from an analysis of the topological structure of a mapping function associated with an admitted surface. The desired properties of a generalized "Streckenkomplex" are inferred from this analysis and these serve as a basis for a definition of this concept. A necessary and sufficient condition that a generalized "Streckenkomplex" represent a simply-connected Riemann surface is established. The relation between the boundary elements of a surface and certain structural aspects of its associated generalized "Streckenkomplex" is given. An analysis is made of the Riemann surfaces associated with $z \sin z$, $z^{-1} \sin z$, $e^z - z^{-1}$. The paper terminates with a comparison between "Streckenkomplex" and generalized "Streckenkomplex" for surfaces for which the former concept is defined.

M. H. Heins (Princeton, N.J.).

Huckemann, Friedrich. Über den Einfluss von Randstellen Riemannscher Flächen auf die Wertverteilung. Math. Z. **65** (1956), 240–282.

The relation between surface structure and the characteristics of an associated mapping function is studied for a certain class $\mathcal{W}$ of simply-connected concrete Riemann surfaces. This class consists of such surfaces satisfying: (1) the ramification points and boundary elements lie over the real axis; (2) the generalized "Streckenkomplex" possesses a finite kernel and a finite number of periodic ends of the following types — logarithmic, simple sine, multiple sine; (3) there is at least one sine end; (4) no sine end has the empty set as kernel; (5) certain regularity conditions. It is shown that the surfaces of $\mathcal{W}$ are parabolic. With a boundary element of an admitted surface there are associated convergence, fusion and growth intensities. The concept of growth intensity leads to a refinement of the Denjoy-Carleman-Ahlfors theorem for the meromorphic functions associated with the admitted surfaces. Defect is related to the convergence and fusion intensities. The treatment of these problems is based upon earlier work of Ahlfors [Acta Math. **58** (1932), 375–406] and Teichmüller's Modulsatz. *M. H. Heins.*

Yang, Tsung-Pan B. Einige Eigenschaften nichtfortsetzbarer, konformer Riemannscher Flächen. Acta Math. Sinica **4** (1954), 301–304. (Chinese. German summary)

Die Randstückmenge einer nichtfortsetzbaren Riemannschen Fläche ist vom absoluten harmonischen Nullmass. *From the author's summary.*

Yang, Tsung-Pan B. Ein elementares Potential auf einer geschlossenen, konformen Riemannschen Fläche, und seine Anwendungen. Acta Math. Sinica **4** (1954), 279–294. (Chinese. German summary)

Der Verfasser macht einen Versuch die bekannte, ebene Potentialtheorie elementaren Teils in die auf einer beliebigen geschlossenen, konformen Riemannschen Fläche einzuordnen. Die Rolle, die früher das Logarithmus spielt, spielt nun das Abelsche Potential dritter Gattung. Auf Grund hierin eingeführten Riemannschen Metrik verallgemeinert man den ersten Nevanlinnaschen Hauptsatz u.s.w. Dabei bleiben einige Fragen ungelöst.

Author's summary.

Yang, Tsung-Pan B. Über einen Existenzsatz. Acta Math. Sinica **4** (1954), 295–299. (Chinese. German summary)

In dieser Note, beweist der Verfasser die Existenz einer eindeutigen, analytischen Funktion auf einem beliebigen Teilgebiet einer geschlossenen, konformen Riemannschen Fläche. Die Beweismethode ist von der Koebeschen verschieden. *Author's summary.*

Kung Sun. On the coefficients of schlicht functions. II. Acta Math. Sinica **6** (1956), 115–125. (Chinese. English summary)

Let $f_k(z)=z+\sum_{n=1}^{\infty} c_{nk+1}^{(k)} z^{nk+1}$ be regular and schlicht in the unit circle $|z|<1$. The chief aim of this note is to give the following result:

$$|c_{2n+1}^{(2)}|<\exp\{-(1-b)^2\}+o(1),$$

where $b=|c_3^{(2)}|$.

From the author's summary

Jenkins, James A. On a result of Keogh. J. London Math. Soc. **31** (1956), 391–399.

Let $f(z)$ be regular univalent and bounded in $|z|<1$ and let $l(\varrho)$ be the length of the image curve under f of the segment $0\leq z\leq\varrho$, $\varrho<1$. Keogh [same J. **29** (1954), 379–382; MR **15**, 862] has proved that as $\varrho\to 1$, $l(\varrho)=o[(\log 1/(1-\varrho))^{\frac{1}{2}}]$ and that if $0<\alpha<\frac{1}{2}$, there is a suitable function and a constant $K>0$ such that

$$l(\varrho)>K(\log 1/(1-\varrho))^{\alpha}$$

as $\varrho\to 1$. The author now improves this lower bound to

$$l(\varrho)>K(\log 1/(1-\varrho))^{\frac{1}{2}}\log\log 1/(1-\varrho),$$

and he asserts that his method will yield still larger orders of growth. *A. W. Goodman* (Princeton, N.J.).

Crum, M. M. A property of schlicht functions. J. London Math. Soc. **31** (1956), 493–494.

Let $\sigma(z)=z+\cdots$ be regular and univalent in $|z|<1$. The author proves that either $\sigma(z)\equiv z$ or there are pairs z_1 and z_2 in $|z|<1$ such that $\sigma(z_1)\sigma(z_2)=1$. This result is well known and has been generalized in many directions. [For details and further references see Goodman, Trans. Amer. Math. Soc. **78** (1955), 82–97; MR **16**, 685.]

A. W. Goodman (Princeton, N.J.).

See also: Huckemann, p. 120; Warschawski, p. 121; Kreyszig, p. 130; Masani, p. 138; Kostyčev, p. 165; Karp and Williams, p. 170.

Geometric Analysis

de Bruijn, N. G. On some multiple integrals involving determinants. J. Indian Math. Soc. (N.S.) **19** (1955), 133–151 (1956).

The main objective of this paper is the discussion of the n-tuple integral Ω whose integrand is $\det \varphi_i(x_j)$, where $\varphi_1, \cdots, \varphi_n$ are functions of x defined for $a\leq x\leq b$, the range of integration being defined by the inequalities $a\leq x_1<\cdots<x_n\leq b$. The basic tool is the signature function $E(x_1, \cdots, x_n)$ defined, for real values of $x_1, \cdots, x_n$ as the product of the functions $\operatorname{sgn}(x_j-x_i)$, $1\leq i<j\leq n$. A multitude of representations for E are derived, in many cases followed by generalizations. We merely state one of the results concerning the evaluation of Ω for the case $n=2m$, m a positive integer: let a_{ij} be the double integral, over the range $a\leq x\leq b$, $a\leq y\leq b$, of the product $\varphi_i(x)\varphi_j(y)\operatorname{sgn}(y-x)$. Then

$$\Omega=\frac{1}{2^m m!}\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}E(j_1, \cdots, j_n)a_{j_1j_2}a_{j_3j_4}\cdots a_{j_{n-1}j_n}.$$

The proof is made by using identities in the theory of Pfaffian forms. *T. Radó* (Columbus, Ohio).

Szekeres, G. A note on the volume of the unitary symmetrical space. J. Indian Math. Soc. (N.S.) **19** (1955), 127–132 (1956).

Let V_n be the space of $n\times n$ complex unitary symmetrical matrices (i.e. the set of all matrices of the form e^{iS}, where S is real symmetrical). V_n is a $\frac{1}{2}n(n+1)$-dimensional subspace in the space of all complex $n\times n$ matrices. The ordinary metric of the full matrix space induces a metric in V_n, and that metric defines the volume vol (V_n) of V_n. The author obtains that $\operatorname{vol} V_n=J_n\cdot\operatorname{vol} O_n$, where

$$J_n=\int\cdots\int\prod_{\mu<\nu}2\sin(\varphi_\mu-\varphi_\nu)\,d\varphi_1\cdots d\varphi_n,$$

integration being taken over the simplex $0\leq\varphi_n\leq\cdots\leq\varphi_1\leq\pi$, and O_n is the space of all real orthogonal $n\times n$ matrices. Its volume has been determined by Hurwitz [cf. Ponting and Potter, Quart. J. Math. Oxford Ser. **20** (1949), 146–154; MR **11**, 75]. The value of J_n was determined by the reviewer [see the paper reviewed above], by non-geometrical methods. The result is that J_n equals twice the volume of the unit n-sphere. A consequence is that $\operatorname{vol}(V_n)=2^{1-\frac{1}{2}n}\operatorname{vol}(O_{n+1})$. It would be desirable to have a direct geometrical proof for these relations, especially since V_n and O_{n+1} have the same dimension.

N. G. de Bruijn (Amsterdam).

Warschawski, S. E. On the distortion in conformal mapping of variable domains. Trans. Amer. Math. Soc. **82** (1956), 300–322.

Let be C a closed Jordan curve in the w-plane, and $f(z)$ a normalized mapping of $|z|<1$ onto the interior of C; the author gives here quantitative information concerning the variations of $f(z)$ and $f'(z)$ under a variation of C. More precisely, he supposes that C and the deformed curve C_1 both possess continuous tangents, whose angles with a fixed direction, respectively denoted by $\tau(s)$, $\tau_1(s)$, as

functions of the arc length, admit the moduli of continuity $\beta(h)$ [resp. $\beta_1(h)$]; C and C_1 are supposed contained in the ring $0<d<|w|<D$, and for both curves, the shorter arc Δs between two points w', w'', must verify $\Delta s/|w'-w''| \leq c$. Further, for almost all results in this work, the author supposes that, to every point $w_1 \in C_1$, there corresponds a point $w \in C$ such that $|w_1-w|<\varepsilon$, and that the angle of tangents at w and w_1 is less than $\eta<\frac{1}{2}\pi$. Under these conditions, if f and f_1 respectively map $|z|<1$ onto the interiors of C, C_1 such that $f(0)=f_1(0)=0$; $f'(0)>0$; $f_1'(0)>0$, there exists a) a constant M_p such that

$$\mathfrak{M}_p(f-f_1)=\left[\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})-f_1(e^{i\theta})|^p\right]^{1/p}\leq M_p\varepsilon,$$

b) a constant K_p such that, if β is convex:

$$\mathfrak{M}_p(f'-f_1')=\frac{1}{2\pi}\left[\int_0^{2\pi}|f'(e^{i\theta})-f_1'(e^{i\theta})|^p\right]^{1/p}\leq K_p\left[\eta+\beta(2cM_p\varepsilon)\right].$$

These results improve a preceding one of the author [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1930**, 344–369]. Some bounds are given for $\mu=\sup|f(z)-f_1(z)|$ and $\mu'=\sup|f'(z)-f_1'(z)|$, under the hypothesis

$$\int_0^t[\beta(x)+\beta_1(x)]x^{-1}dx<\infty;$$

similarly, if the curves are conseniently represented in parametric form by $w(t)$ and $w_1(t)$, there exists a constant N such that the derivatives of $\varphi=f^{-1}$ and $\varphi_1=f_1^{-1}$ satisfy $|\varphi'[w(t)]-\varphi_1'[w_1(t)]|\leq N\varepsilon$.

Some improvements are given, under stronger hypotheses, and an application of these results to the estimation of the neglected term in Hadamard's variation formula for Green's function.

It must be noted that all the given bounds, like M_p, K_p, N, only depend on a finite number of parameters such as d, D, c, $\cdots$, η and in no other way on C and C_1.

J. Lelong (Paris).

Ivanov, V. K. On the solubility of the inverse problem of the logarithmic potential in finite form. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 598–599. (Russian)

The inverse problem referred to in the title is: given an external potential function $V(x, y)=V(z)$, to find a domain D having this potential when filled with matter of unit density. The author considers the case where

$$u(z)=-\frac{2}{\pi}\frac{\partial V}{\partial z}=\sum_{n=0}^{\infty}\frac{c_n}{z^{n+1}}$$

is a rational function of z. Suppose the problem has a solution D with $z=0$ as an interior point; let $z(t)$ be a regular function mapping $|t|<1$ conformally on D so that $z(0)=0$, and write

$$z^*(t)=\overline{z(1/t)}.$$

The function $z^*(t)$ then satisfies the integral equation

$$(1)\qquad z^*(t)=-\frac{1}{2i\pi}\int_\Gamma\frac{u[z(\tau)]}{\tau-t}\,d\tau\quad(|t|>1),$$

where Γ is the unit circle in the τ-plane. If $u(z)$ is rational, having poles at z_1, $\cdots$, z_s with multiplicities k_1, $\cdots$, k_s, and $k_1+\cdots+k_s=n$, it follows that

$$z^*(t)=\frac{p(t)}{\prod_{i=1}^{s}(t-t_i)^{k_i}},$$

where $p(t)$ is a polynomial of degree $n-1$, and $z_i=z(t_i)$ $(1\leq i\leq s)$. This expression contains $n+s$ complex parameters, which the integral equation (1) can be used to determine.

F. Smithies (Cambridge, England).

Tsuji, Masatsugu. Littlewood's theorem on subharmonic functions in a unit circle. Comment. Math. Univ. St. Paul. **5** (1956), 3–16.

Let $u(z)$ be subharmonic in $|z|<1$, such that

$$\int_0^{2\pi}|u(re^{i\Theta})|d\Theta=O(1)$$

for $r\to 1$. A theorem of J. E. Littlewood [Proc. London Math. Soc. (2) **28** (1928), 383–394] states that $u(e^{i\Theta})=\lim_{r\to 1}u(re^{i\Theta})$ for almost all Θ $(0\leq\Theta<2\pi)$. Under the additional hypothesis that $\mu(|z|<r)=O[(1-r)^{-\lambda}]$ for some $\lambda<1$ (μ=measure associated with u) the author now proves that, for almost all Θ, the following is true:

$$\lim_{t\to 0}u(e^{i\Theta}-te^{i(\Theta+\Psi)})=u(e^{i\Theta})$$

for almost all Ψ $(|\Psi|<\frac{1}{2}\pi)$.

A. Huber (Zurich).

Shah, Tao-Shing. On the mapping radii and Green functions of non-overlapping domains. Acta Math. Sinica **6** (1956), 13–18. (Chinese. English summary)

Let G_1, G_2 be two non-overlapping domains contained in certain domain G. Let $R(a_i, G_i)$ be the mapping radius of domain G_i with respect to $a_i \in G_i$, i.e. the reciprocal of capacity constant of G_i with respect to a_i. Let $g(z, \zeta)$ be the Green function of G with a logarithmic pole at ζ, and $h(z, \zeta)$ be the regular part of $g(z, \zeta)$, i.e.

$$g(z, \zeta)=\log\frac{1}{|z-\zeta|}+h(z, \zeta).$$

In the present note, we prove that, for $x_1>0$, $x_2>0$,

$$\overline{Bd}_{G_1+G_2\subset G}R(a_1, G_1)^{x_1^2}R(a_2, G_2)^{x_1^2}=|a_1-a_2|^{2x_1x_2}\exp\Big(\sum_{i,j=1}^{2}(-1)^{i+j}h(a_i, a_j)x_ix_j\Big),$$

by the method of extremal length.

Author's summary.

See also: Rado and Reichelderfer, p. 115; Kuramochi, p. 120; Cimmino, p. 131; Hristoforov, p. 163; Pivko, p. 168; Smythe, p. 169; Petrašen', Ivanova and Vol'f, p. 175.

Special Functions

Heselden, G. P. M. Some inequalities satisfied by incomplete beta functions. Skand. Aktuarietidskr. **38** (1955), 192–200.

Let

$$I_x(p, q)=B_x(p, q)/B_1(p, q)\ (0\leq x\leq 1,\ p>0,\ q>0),$$

where $B_x(p, q)=\int_0^x t^{p-1}(1-t)^{q-1}dt$ is the incomplete beta function. For $n>2$, $x>-(n-1)/n^{\frac{1}{2}}$, let

$$S(x, n)=((n-1)/n^{\frac{1}{2}})[I_t(\tfrac{1}{2}(n-2), \tfrac{1}{2}(n-2))-I_t(\tfrac{1}{2}n, \tfrac{1}{2}(n-2))],$$

$$R(x, n)=1-I_t(\tfrac{1}{2}(n-2), \tfrac{1}{2}(n-2)),$$

$$T(x, n)=xR(x, n)+S(x, n),$$

where $t=t(x, n)=\max(0, \frac{1}{2}-xn^{\frac{1}{2}}/2(n-1))$.

The author shows that

(A) $$\lim T(x,n)=(2\pi)^{-\frac{1}{2}}\{x\int_{-x}^{\infty}e^{-w^2}dw+e^{-x^2}\},$$

(B) $$S(x,n)\leqq S(x,n+1),$$

(C) $$T(x,n)\leqq T(x,n+1).$$

P. G. Rooney (Toronto, Ont.).

Meĭman, N. N. On recurrence formulas for power sums of the zeros of Bessel functions. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 190–193. (Russian)

Let $\sigma_\nu^{(m)}=\sum_{j=-\infty}^{\infty}\lambda_{\nu,j}^{-m}$, where $\lambda_{\nu,j}$ are the non-vanishing zeros of $J_\nu(z)$. Then

$$2(\nu+k+1)\sigma_\nu^{(2k+2)}=\sum_{i=1}^{k}\sigma_\nu^{(2i)}\sigma_\nu^{(2k+2-2i)},$$

where $2(\nu+1)\sigma_\nu^{(2)}=1$. Similar formulae are obtained for the non-vanishing zeros of $zJ_\nu'(z)-HJ_\nu(z)$.

R. G. Langebartel (Urbana, Ill.).

Blanuša, Danilo. Une espèce de théorèmes intégraux des fonctions de Bessel. Bull. Internat. Acad. Yougoslave Sci. Beaux-Arts (N.S.) **6** (1952), 5–58.

A translation from the article reviewed in MR **14**, 979.

Arscott, F. M. Recurrence formulae for Lamé polynomials. J. London Math. Soc. **31** (1956), 360–364.

L'auteur, ayant remarqué que des relations de récurrence existent entre les fonctions de Mathieu de différents ordres, s'est demandé s'il était possible de former des relations analogues avec les fonctions de Lamé. Il donne ici une méthode qui s'applique seulement aux polynomes de Lamé, pour lesquels le résultat est d'ailleurs extrêmement intéressant. Les relations, contrairement à ce qui se passe pour les fonctions de Mathieu ne contiennent, en effet, qu'un nombre fini de termes. $Ep_n^m(\alpha,\beta)$ désignant le produit $E_n^m(\alpha)E_n^m(\beta)$ (E_n^m étant lui-même un polynome de Lamé) la relation d'orthogonalité suivante est connue:

$$\int_{-2K}^{+2K}\int_{K-2iK'}^{K+2iK'}(\mathrm{sn}^2\,\alpha-\mathrm{sn}^2\,\beta)Ep_n^m(\alpha,\beta)Ep_{n'}^{m'}(\alpha,\beta)d\alpha d\beta=$$

$$=\begin{cases}0 \text{ si } n\neq n',\ m\neq m'\\ \pm i \text{ si } n=n',\ m=m'\end{cases}$$

[cf. Hobson, The theory of spherical and ellipsoïdal harmonics, Cambridge, 1931]. Par des considérations très simples d'algèbre linéaire et de dénombrement de coefficients, relativement à l'expression possible de $Ep_{2n}^m(\alpha,\beta)$ par exemple, sous la forme d'une somme telle que $\sum_{q=0}^{n}\sum_{r=0}^{q}A_q^rEp_{2q+1}^r(\alpha,\beta)$ l'auteur trouve, très simplement, des relations entre les $Ep_n^m(\alpha,\beta)$, d'où il tire des relations entre les polynomes de Lamé eux-mêmes en y faisant β=constante.

La méthode s'applique aussi bien pour trouver des relations entre des polynomes de Lamé de types différents qu'entre des polynomes de même type, et laisse ouverte l'intéressante question de savoir si les autres solutions de l'équation de Lamé satisfont aux mêmes formules.

R. Campbell (Caen).

Felsen, Leopold B. Some definite integrals involving conical functions. J. Math. Phys. **35** (1956), 177–178.

Several integrals of the type

$$\int_0^\infty f(x)P^m_{ix-\frac{1}{2}}(\cos\theta)P^m_{ix-\frac{1}{2}}(\cos\theta')\,dx$$

are evaluated. *N. D. Kazarinoff.*

Fabian, William. A generalised hypergeometric function. Proc. Edinburgh Math. Soc. (2) **9** (1956), 151–153.

The class of functions of the form

$$\frac{\Gamma(c)\Gamma(\lambda+1)}{\Gamma(a)\Gamma(b)}\sum_{n=-\infty}^{n=\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(\lambda+n+1)}z^{\lambda+n},$$

λ, a, b, c, and z complex, is discussed. The author shows that such functions may be represented as simple combinations of powers of z and of derivatives of integrals involving ordinary ${}_2F_1$'s. *N. D. Kazarinoff.*

See also: Richert, p. 123; Geissler, p. 151; Kennemann, p. 153; Clarke, p. 157; Plesset and Mitchell, p. 168; Moshinsky, p. 168; Kogan, p. 169; Wilhelmsson, p. 170; Petrašen′, Ivanova and Vol′f, p. 175.

Sequences, Series, Summability

Moore, P. G. The geometric, logarithmic and discrete Pareto forms of series. J. Inst. Actuar. **82** (1956), 130–136.

An expository paper. *Z. W. Birnbaum.*

Clunie, J. On a certain series of Abel. Proc. Edinburgh Math. Soc. (2) **9** (1956), 132–144.

This paper begins with historical remarks to the effect that if $f(z)$ has an appropriate integral representation, then the series

$$\sum_{n=0}^{\infty}\frac{z(z-n)^{n-1}}{n!}f^{(n)}(n)$$

converges to $f(z)$ for each z for which some manipulations can be justified. It is known and easy to see that if $f(z)=(1+z)^{-1}$, then the series converges everywhere to an entire function $G(z)$ which can be equal to $f(z)$ only for isolated values of z. An asymptotic expansion of $G(z)$ is obtained, and it is shown that in a particular region of the plane $G(z)$ is asymptotically equal to $(1+z)^{-1}$.

R. P. Agnew (Ithaca, N.Y.).

Richert, Hans-Egon. Beiträge zur Summierbarkeit Dirichletscher Reihen mit Anwendungen auf die Zahlentheorie. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. 1956, 77–125.

The paper is devoted to the problem of obtaining information about the convergence or Riesz summability of a Dirichlet series from the function-theoretic behaviour of its sum, in particular from its order of growth on vertical lines. The author derives a large number of new results, and compares their strength to existing results. Owing to the large number of results quoted, the paper can also be used as a literature report, though it was not written for that purpose. The main feature of the paper is that old theorems dealing with the Lindelöf function $\mu(\sigma)$ are replaced by the stronger theorems involving the Carlson function $\mu_1(\sigma)$ (see below), whereas new results involving $\mu_2(\sigma)$ are obtained by introduction of a new type of strong summability. Moreover, some results are strengthened by admitting a finite number of poles.

Let $Z(s)=Z(\sigma+it)$ be analytic in a half-plane $\sigma>E$ apart from a finite number of poles in a strip $E<\sigma\leqq S$. Assume that $Z(\sigma+it)=O(|t|^C)$ $(\sigma>E,\ |t|>T_0)$ for some constant C. Further, it is assumed that $Z(s)$ is the sum of the Dirichlet series $\sum_1^\infty a_n n^{-s}$ if σ is sufficiently large. σ^* denotes the inf of the set of all ξ such that $a_n=O(n^\xi)$.

If $\kappa \geqq 0$, then σ_κ is the inf of the set of all ξ such that the series is summable by Riesz means $(R, \log n, \kappa)$ at $s=\xi$, and similarly $\bar{\sigma}_\kappa$ refers to absolute summability by these means (so σ and $\bar{\sigma}_0$ denote the abscissae of convergence and of absolute convergence, respectively). If $\sigma > E$, Carlson's function $\mu_k(\sigma)$ is defined as the inf of all numbers ξ satisfying $\int_{T_0<|t|<T} |Z(\sigma+it)|^k dt = O(T^{\xi+1})$ (where T_0 is suitably chosen in order to avoid the possible poles). The author proves that (1) $\sigma^* \leqq \eta + \mu_1(\eta)$ $(\eta > E)$.

$$(2)\quad \sigma_\kappa \leqq \sigma^* + 1 - (\kappa+1)(\sigma^*+1-\eta)/(1+M) \leqq \eta + M - \kappa \quad (\eta > S)$$

where $M = \max(\mu_1(\eta), \kappa)$, and related results involving residues if $\eta > E$. (3) $\bar{\sigma}_0 \leqq \eta + \frac{1}{2}\mu_2(\eta) + \frac{1}{2}$ $(\eta > E)$.

Next the author develops a theory about strong summability $|R, \log n, \kappa|^2$: the series $\sum_0^\infty c_n$ is summable to c by this method if

$$\int_1^x |C_\kappa(u) - c \log^\kappa u|^2 u^{-1} du = o(\log^{2\kappa+1} x),$$

where $C_\kappa(x) = \sum_{n \leq x} c_n$. If $\kappa' > \kappa \geqq 0$ it is shown that summability $(R, \log n, \kappa)$ implies summability $|R, \log n, \kappa|^2$, and summability $|R, \log n, \kappa|^2$ implies summability $|R, \log n, \kappa'|^2$. If a Dirichlet series is summable $|R, \log n, \kappa|^2$ at a point $s=\sigma'+it'$, then the same thing applies to any point $\sigma+it$ $(\sigma > \sigma')$. Therefore it is possible to define the abscissa $\tilde{\sigma}_\kappa$ of summability $|R, \log n, \kappa|^2$. It is shown that $\bar{\sigma}_{\kappa+1} \leqq \tilde{\sigma}_\kappa \leqq \frac{1}{2}(\bar{\sigma}_{\kappa+1} + \sigma_\kappa) \leqq \sigma_\kappa$. There is also a product theorem: If $\kappa \geqq 0$, $\lambda \geqq 0$, and if the series $Z_1(s)$ and $Z_2(s)$ are summable $|R, \log n, \kappa|^2$ and $|R, \log n, \lambda|^2$, respectively, at $s=0$, then the product series $Z_1(s)Z_2(s)$ is summable $(R, \log n, \kappa+\lambda+1)$ at $s=0$.

Bohr showed in his dissertation [Collected mathematical works, København, 1952, v. I, Dansk Matematisk Forening, København, 1952; MR **15**, 276] that the abscissa σ_0 of ordinary convergence is not uniquely determined by the Lindelöf μ-function. It is now shown, however, that there is a complete correspondence between μ_2 and $\tilde{\sigma}_\kappa$: If $\kappa \geqq 0$, then $\tilde{\sigma}_\kappa$ is the point where $\mu_2(\tilde{\sigma}_\kappa) = 2\kappa+1$, as long as this point lies to the right of S. (In particular $\mu_2(\tilde{\sigma}_0) = 1$, and in this connection it may be remarked that in special cases a proof of the stronger statement $\mu_2(\sigma_0) = 1$ would imply some well-known conjectures like the one concerning the exponent $\frac{1}{4}+\varepsilon$ in the problem of lattice points inside a circle.) The author extends the function $\tilde{\sigma}_\kappa$ to the interval $\kappa \geqq -\frac{1}{2}$ by the equation $\mu_2(\tilde{\sigma}_\kappa) = 2\kappa+1$. He shows that (1) $\sigma_\kappa \leqq \tilde{\sigma}_{\kappa-\frac{1}{2}} \leqq \bar{\sigma}_\kappa$ $(\kappa \leqq 0)$; (2) $\bar{\sigma}_{\kappa+1} \leqq \tilde{\sigma}_\kappa$ $(\kappa \geqq -\frac{1}{2})$; (3) $\tilde{\sigma}_\kappa$ is a concave function of κ $(\kappa \geqq -\frac{1}{2})$; (4) $\sigma_\kappa \leqq \tilde{\sigma}_\kappa + \frac{1}{2} \leqq \sigma_{k+1} + 1$ $(\kappa \geqq 0)$. (5) $\bar{\sigma}_\kappa \leqq \tilde{\sigma}_{\kappa-\frac{1}{2}} + \frac{1}{2} \leqq \sigma_\kappa + 1$ $(\kappa \geqq 0)$.

The results of the paper are applied in a detailed discussion of the generalized Piltz divisor problem about the error term in the asymptotic formula of

$$f(x) = \sum_{n_1 \cdots n_k \leq x} \chi_1(n_1) \cdots \chi_k(n_k),$$

whose main term equals the residue of

$$s^{-1} x^s L(s, \chi_1) \cdots L(s, \chi_k)$$

at $s=1$. Further applications concern Hecke series related to cusp forms.

N. G. de Bruijn (Amsterdam).

Gallie, T. M., Jr. Region of convergence of Dirichlet series with complex exponents. Proc. Amer. Math. Soc. **7** (1956), 627–629.

Given the Dirichlet series $\sum c_n e^{-\lambda_n s}$ where the exponents λ_n are complex, the author constructs the region $D=[s; d(s) \geqq 0]$ where $d(s) = \liminf d(n, s)$ and $d(n, s) = -|\lambda_n|^{-1} \log |c_n e^{-\lambda_n s}|$. It is shown that D is the maximal region of absolute convergence of the series and that the latter is absolutely convergent in the domain

$$D_L = [s; d(s) > L]$$

where $L = \limsup (\log n)|\lambda_n|^{-1}$; $d(s)$ is the distance of s from the boundary of D. [Cf. the reviewer, Ann. of Math. (2) **25** (1924), 261–278, where the case $L=0$ occurs and a more complicated determination of D is given.]

E. Hille (New Haven, Conn.).

Alexits, G. Ein Summationssatz für Orthogonalreihen. Acta Math. Acad. Sci. Hungar. **7** (1956), 5–9. (Russian summary)

Let $\{\varphi_n(x)\}$ form an orthonormal system in $\langle a, b \rangle$. It is shown that the series $\sum c_n \varphi_n(x)$ is (C, α)-summable p.p., for every $\alpha > 0$, if $c_n = O(q_n)$ where $q_n \searrow$ and $\sum q_n/\sqrt{n} < \infty$. This condition does not imply the known condition $\sum c_n^2 (\log \log n)^2 < \infty$, nor vice versa.

W. W. Rogosinski (Newcastle-upon-Tyne).

Włodarski, L. Propriétés des méthodes continues de limitation du type de Borel. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 173–175.

Let k be a nonnegative integer and let $\alpha = 2^{-k}$. A sequence $\xi_0, \xi_1, \xi_2, \cdots$ is said to be evaluable B_k to L_k if the series in

$$F_k(t) = \alpha e^{-t} \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n - 1)} \xi_n$$

converges for each $t > 0$ and $F_k(t) \to L_k$ as $t \to \infty$. The methods B_k are all regular, and B_0 is the Borel exponential method. For each $k=0, 1, 2, \cdots$ there exists a sequence evaluable B_k but nonevaluable B_q when q is a nonnegative integer for which $q \neq k$. If $0 \leq p < q$ and a given sequence is evaluable B_p, then it is also evaluable B_q to the same value provided the B_q transform exists. If a given sequence is evaluable B_k for at least one k, then it is also evaluable to the same value by the Abel (or Euler or Poisson) power series method provided the Abel transform exists. These results imply that the methods B_k are consistent with each other and with the methods of Abel, Cesàro, and Euler. When z is a complex number which is not both real and 1 or more, the geometric series $\sum z^k$ is evaluable B_k to $1/(1-z)$ for at least one k. There are no proofs and it is promised that proofs will be published elsewhere.

R. P. Agnew (Ithaca, N.Y.).

Włodarski, L. Sur la concordance entre les méthodes intégrales de sommation du type de Borel. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 177–178.

Let $\alpha > 0$. A series $\beta_0 + \beta_1 + \cdots$ is evaluable B_α^* to L_α if

$$\int_0^\infty e^{-t} \Big[\sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n - 1)} \beta_n \Big] dt = L_\alpha.$$

The methods B_α^* for which α has the form $1, 2^{-1}, 2^{-2}, 2^{-3}, \cdots$ are consistent with each other, with the methods B_k of the paper reviewed above, and with the methods of Abel, Cesàro, and Euler. There are no proofs, and it is promised that the proofs will be published elsewhere.

R. P. Agnew (Ithaca, N.Y.).

See also: Freĭman, p. 112; Rechard, p. 119; Campbell, p. 125.

Approximations, Orthogonal Functions

Jecklin, H. Varia zur hyperbolischen Interpolation von Reservekurven. Mitt. Verein. Schweiz. Versich.-Math. **56** (1956), 49–63.

The increase of the value of a life assurance, especially of an endowment, is better approximated by a hyperbola than by a polynomial. An expression of the form

$$_tV=t\cdot\frac{c}{n}+\frac{t}{A+Bt}$$

may even under special conditions give the exact values.

Interpolation for the values at intermediate times is illustrated by numerical examples. The method may lead to a group valuation method using three status numbers.

P. Johansen (Copenhagen).

Krasnoselski, M. A. Two remarks on the method of successive approximations. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat. Fiz. (3) **10** (1956), no. 2(17), 55–59. (Romanian)

A translation of the article listed in MR **16**, 833.

★ **Helman, D. Tchebycheff approximations for amplitude and delay with rational functions.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 385–402. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1956.

Filters with equal-ripple transmission characterization in both the pass band and the stop band are synthesized by the use of Tchebycheff rational functions. The method is algebraic and the computations can be carried out to any desired degree of accuracy.

R. J. Duffin (Pittsburgh, Pa.).

Campbell, Robert. Détermination effective de toutes les moyennes de Cesàro d'ordre entier pour des séries de polynomes orthogonaux comprenant ceux de Laguerre et de Hermite. C. R. Acad. Sci. Paris **243** (1956), 882–885.

Let $\mathcal{F}$ be the set of all families of orthogonal polynomials satisfying relations of the form

$$P_n(x)=(A_nx+B_n)P_{n-1}(x)-C_nP_{n-2}(x),$$

$$P_n(x)=f_n(x)P_{n-1}(x)-g_n(x)P_{n-1}'(x),$$

where $2f_nA_n^{-1}-(x+B_nA_n^{-1})$ and g_n/A_n are independent of n. A scheme is presented for calculating the means (C, k) for any element of $\mathcal{F}$. Application is made to the polynomials of Hermite and Laguerre.

N. D. Kazarinoff (Ann Arbor, Mich.).

Bade, William G.; and Schwartz, Jacob T. On Mautner's eigenfunction expansion. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 519–525.

Let (S, Σ, ν) be a σ-finite measure space and T a self-adjoint operator on $L_2(S, \Sigma, \nu)$. The authors study conditions for the existence of an eigenfunction expansion. A bounded linear operator A on $L_2(S, \Sigma, \sigma)$ is said to have the property (α) if there exists a sequence of sets S_n of finite measure covering S such that ν-ess $\sup_{S_n}|A(f)(s)|<\infty$ for each $f\in L_2(S, \Sigma, \nu)$. One of the main theorems of the paper (Theorem 1.2) asserts the following. Let T be a self-adjoint operator on $L_2(S, \Sigma, \nu)$ with the spectral resolution $E(\lambda)$. If for each bounded Borel subset e of the real line the projection $E(e)$ has the property (α) then T has an eigenfunction expansion. It is assumed that (S, Σ, ν) is σ-finite, but not that $L_2(S, \Sigma, \nu)$ is separable. Variuor consequences are derived from this theorem, in particulas the result of the reviewer [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 49–53; MR **14**, 659] that if $(T-cI)^{-1}$ is an integral operator of the Carleman type, then T has an eigenfunction expansion. The authors state that Garding's generalization of the reviewers result [Applications of the theory of direct integrals of Hilbert spaces to some integral and differential operators, Inst. Fluid Dynamics Appl. Math., Lecture series no. 11, Univ. of Maryland, 1954; MR **17**, 159] follows from a straightforward generalization of their result.

The results of the present paper are applied to yield the existence of an eigenfunction expansion for elliptic differential operators, but also Hörmander's result [Acta Math. **94** (1955), 161–248; MR **17**, 853] on certain non-elliptic partial differential operators with constant coefficients.

Next it is shown that when $L_2(S, \Sigma, \nu)$ is separable, then the hypotheses of Theorem 1.2 are actually equivalent to the requirement that some bounded function of T be an integral operator of the Carleman type. This is shown by giving a simple characterization of operators of the Carleman type.

The paper closes with results on the linear independence of certain kernels and spectral multiplicity. More details are promised in a forthcoming book.

{The reviewer is wondering whether it is now possible to attack the important general problem of eigenfunction expansions of differential operators: Given a self-adjoint extension of a partial differential operator; is it possible to give simple necessary and sufficient conditions for the existence of an eigenfunction expansion?}

F. I. Mautner (Baltimore, Md.).

Tornheim, Leonard. Approximation by families of functions. Proc. Amer. Math. Soc. **7** (1956), 641–643.

Let F be a n-parameter family of single-valued real continuous functions $f(x)$ on an interval $a\leqq x\leqq b$ such that for every set of points (x_i, y_i), $i=1, \cdots, n$ with $a\leqq x_1<\cdots<x_n\leqq b$ there is exactly one $f(x)$ with $f(x_i)=y_i$. The author proceeds with his earlier investigation about the best approximations in F [Trans. Amer. Math. Soc. **69** (1950), 457–467; MR **12**, 395]. For $k\geqq 1$ the modulus of approximation $M^{(k)}$ of g by f is defined as $M^{(k)}(f)=\int_a^b|f-g|^k dx$ $(k<\infty)$ and $M^{\infty}(f)=\max|f-g|$. The best k-approximant in F of continuous function g is a function f_0 in F for which $M^{(k)}(f_0)=\inf M^{(k)}(f)$, $f\in F$. Following results are proved. The best k-approximant always exists and it is unique if F is linear and $k>1$. If $n=1$ a best k-approximant need not be unique for $k\neq\infty$. Let f_k be a sequence of best k-approximants, then $\lim_{k\to\infty}f_k=f_\infty$.

M. Tomić (Beograd).

Mairhuber, John C. On Haar's theorem concerning Chebychev approximation problems having unique solutions. Proc. Amer. Math. Soc. **7** (1956), 609–615.

Die Frage, zu welchen kompakten und mindestens $n\geqq 2$ Punkte enthaltenden Punktmengen M des E_k stetige (reelle) Funktionen $f_1(x), \cdots, f_n(x)$, $x\in M$ existieren, für die

$$\min_{a_1,\cdots,a_n}\max_{x\in M}|f(x)-\sum_1^n a_\nu f_\nu(x)|$$

für jedes stetige (reelle) $f(x)$ eine eindeutige Lösung $(a_1^*, \cdots, a_n^*)$ besitzt, wird beantwortet durch die notwendige und hinreichende Bedingung: M ist homöophorm

zu einer abgeschlossenen Untermenge des Umfanges eines Kreises. Der zentrale Teil des Beweises besteht in dem Nachweis, dass jede kompakte Menge N im E_n, die von jeder Hyperebene in höchstens n Punkten geschnitten wird, einer abgeschlossenen Untermenge des Umfanges eines Kreises homöomorph ist. Dieser Sachverhalt wird angewandt auf den Satz von A. Haar [Math. Ann. **78** (1918), 294–311], nach dem die Eindeutigkeitsforderung gleichwertig ist mit dem Nichtverschwinden der Determinante $|f_i(x_j)|$ $(i, j=1, \cdots, n)$ für je n verschiedene Punkte $x_1, \cdots, x_n$ aus M. *A. Peyerimhoff.*

Atkinson, F. V. On polynomials with least weighted maximum. Proc. Amer. Math. Soc. **7** (1956), 267–270.

Le poids $w(x)$ continu positif étant donné sur $[a, b]$, on considère le polynome $p_n(x)=x^n+\cdots$ minimisant le norme $|wp_n|$ dans $C(a, b)$. L'auteur établit que les zéros de p_n séparent ceux de p_{n-1}. Un résultat antérieur de l'auteur [Arch. Math. **3** (1952), 83–86; MR **14**, 269] concerne le même problème dans $L^p(a, b)$ $(p>1)$.

J. P. Kahane (Montpellier).

See also: Arscott, p. 123; Alexits, p. 124; Owens, p. 132; Bakel'man, p. 147; Onoyama, p. 156.

Trigonometric Series and Integrals

Yüh, Ming-I. Estimation of a trigonometric sum. Acta Math. Sinica **6** (1956), 105–114. (Chinese. English summary)

The object of this note is to prove the following theorem. Let $\varepsilon>0$ be any given number, $N>2$, $r=\log N$, $1<H\leqq e^r$, $\tau=NH^{-1}$,

$$\alpha=\frac{a}{q}+z;\ (a, q)=1;\ 0<q\leqq e^{r\varepsilon/2},\ |z|\leqq\frac{1}{q\tau},$$

$$S=\sum_{\substack{p+z=p'\\ p\leqq N}} e^{2\pi i\alpha p},$$

where p and p' denote primes, then

$$S\ll\frac{N(rq)^\varepsilon}{r^2\sqrt{q}}.$$

An application of this theorem will appear later.

Author's summary.

Yüh, Ming-I. A note on the expressions and estimations of a kind of trigonometrical sums. Acta Math. Sinica **6** (1956), 35–54. (Chinese. English summary)

Skvorcov, P. G. On strong convergence of de La Valleé Poussin sums in Orlicz space. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 774–776. (Russian)

Consider the Orlicz space L^M of all measurable functions $f(x)$ on $(0, 2\pi)$, where the function $M(u)$ satisfies the condition $M(2u)=o\{\mu(u)\}$ for $u\to\infty$ [for the definition of L^M see Orlicz, Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. **1932**, 207–220; or A. Zygmund, Trigononometrical series, Warszawa-Lwów, 1935, esp. p. 95 et seq.]. Let $s_n(f)$ be the partial sums of the Fourier series of f, and

$$\sigma_{n,m}(f)=\{s_{n-m}(f)+s_{n-m+1}(f)+\cdots+s_n(f)\}/(m+1)$$

the delayed arithmetic means of $\{s_n(f)\}$; we suppose that $m=m(n)=o(n)$. The author shows that if for all $f\in L^M$ we have $\lim\|f-\sigma_{n,m}(f)\|_M=0$ then necessarily

$$\liminf_{u\to+\infty} M(2u)/M(u)>2.$$

A. Zygmund (Chicago, Ill.).

Mohanty, R.; and Mohapatra, S. On the absolute logarithmic summability of a Fourier series. Math. Z. **65** (1956), 207–213.

Stečkin, S. B. On absolute convergence of Fourier series. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 385–412. (Russian)

A sequence of positive integers $n_1<n_2<\cdots$ is said to belong to the class $\mathfrak{R}$ if there is an $A>1$ such that for any integer $n>0$ the number of representations of n in the form

$$n=\pm n_{k_1}\pm n_{k_2}\pm\cdots\pm n_{k_s}\ (k_1<k_2<\cdots<k_s)$$

does not exceed A^s $(s=1, 2, \cdots)$. A sequence which is the union of a finite number of sequences of the class $\mathfrak{R}$ is said to be of the class $\mathfrak{R}_\sigma$. The main result of the paper is that if $\{n_k\}$ is of the class $\mathfrak{R}_\sigma$, and if

$$(*)\qquad \sum (a_{n_k}\cos n_kx+b_{n_k}\sin n_kx)$$

is the Fourier series of a function bounded above (or below), then $\sum(|a_{n_k}|+|b_{n_k}|)$ is finite. This generalizes a previous result of Sidon [Acta Univ. Szeged. Sect. Sci. Math. **10** (1943), 206–253; MR **8**, 150] who made a stronger assumption about $\{n_k\}$, namely that it is the union of a finite number of lacunary (in the sense of Hadamard) sequences. Another result of the present paper is that if $\{n_k\}$ is the union of a finite number of lacunary sequences, and (*) is the Fourier series of a continuous function f, then the ratio of any two numbers $E_n(f)$, $R_n(f)$, $A_n(f)$ is contained between two positive numbers independent of n, where $E_n(f)$ is the best approximation of f by trigonometric polynomials of order n, and

$$R_n(f)=\max_n\Big|\sum_{n_k>n}(a_{n_k}\cos n_kx+b_{n_k}\sin n_kx)\Big|,$$

$$A_n(f)=\sum_{n_k>n}(|a_{n_k}|+|b_{n_k}|).$$

A. Zygmund (Chicago, Ill.).

Putnam, C. R. A note on inverses of differential operators. Math. Z. **64** (1956), 149–150.

Let $L_jX=-X''-q_jX$, $j=1, 2$, where $q_j(t)$ are continuous and real on $0\leqq t<\infty$. Suppose L_1 is in the limit point case and that $q_2(t)-q_1(t)\to0$ as $t\to\infty$. Then L_2 is in the limit point case and the author proves that the cluster spectra of L_1 and L_2 are identical. *N. Levinson.*

Matsuyama, Noboru. On the convergence of some gap series. Sci. Rep. Kanazawa Univ. **3** (1955), no. 1, 11–20.

Cette note a pour l'objet l'étude des séries lacunaires du type (1) $\sum_1^\infty c_n\varphi(\lambda_nx)$, où $\varphi(x+2\pi)=\varphi(x)$, $\int_0^{2\pi}\varphi(x)dx=0$, $\varphi(x)\in\mathrm{Lip}\,\alpha$, $0<\alpha<1$. En continuant les recherches de Kac [Ann. of Math. (2) **44** (1943), 411–415; MR **5**, 4] et Hartman [Duke Math. J. **9** (1942), 404–405; MR **4**, 39], Kawata et Udagawa [Kōdai Math. Rep. no. **5–6** (1949), 19–22; MR **12**, 175] a obtenu plusieurs résultats relatifs à la série (1), où $\{\lambda_n\}$ n'est pas nécessairement une suite des entiers. L'auteur a donné quelques extensions de ces derniers résultats. Ainsi, par exemple, en posant $1\leqq n(x)\leqq N$, $\omega(n)\uparrow\infty$, $n\omega(n)=o(1)$, $\lambda_{n+1}/\lambda_n\geqq\lambda>1$, on a pour les sommes

partielles de (1), l'inégalité suivante

$$\left|\int_{-\infty}^{\infty} S_n(x)d\left(\frac{\sin^2 x/2}{2\pi x^2}\right)\right| \leqq A\left(\sum_{n=1}^{\infty} c_n^2\omega(n)\right)^{\frac{1}{2}}.$$

M. Tomić (Beograd).

See also: Maurin, p. 130; Masani, p. 138; Sretenskiĭ, p. 165.

Integral Transforms

★**Tranter, C. J. Integral transforms in mathematical physics. 2nd ed.** Methuen and Co. Ltd., London; John Wiley and Sons, Inc., New York, 1956. ix+133 pp. $2.00.

This is a second edition. The first edition was published in 1951 [MR 13, 127]. The present edition contains, in an additional chapter, an account of recent work on dual integral equations.

See also: Sebastião e Silva, p. 137; Shack, p. 169; Papoulis, p. 171.

Ordinary Differential Equations

Chin, Yuan-Shun. Theory of regional analysis of ordinary differential equations. I. Acta Math. Sinica 6 (1956), 19–34. (Chinese. English summary)

Consider a system of ordinary differential equations

$$(1)\qquad \frac{dx}{dt}=X(x, y), \frac{dy}{dt}=Y(x, y),$$

where $X(x, y)$ and $Y(x, y)$ possess partial derivatives of first and second orders in a closed region S. To approach the system (1) we take the following system

$$(2)\qquad \frac{dx}{dt}=X_n(x, y), \quad \frac{dy}{dt}=Y_n(x, y),$$

where $X_n(x, y)$ and $Y_n(x, y)$ are polynomials of degree n. In order to determine $X_n(x, y)$ and $Y_n(x, y)$, two conditions will be imposed on. We call the first one the condition of expressibility, i.e.

$$(3)\qquad \frac{\partial X_n}{\partial x}+\frac{\partial Y_n}{\partial y}\equiv 0.$$

And the second one is the condition of approximation, the functional

$$(4)\qquad I_n(X, Y; S)\equiv \iint_S \{(X-X_n)^2+(Y-Y_n)^2\}dS$$

assuming the absolute minimum under the condition (3).

The following general results are proved. (i) The necessary and sufficient condition that such pair of functions $X_n(x, y)$ and $Y_n(x, y)$ uniquely exist is that

$$(5)\qquad H_n(X, Y, S)\equiv \sum_{i=0}^{n}\left[\iint_S Xy^i dS\right]^2+\sum_{i=0}^{n}\left[\iint_S Yx^i dS\right]^2$$
$$+\sum_{i\geq 0, j\geq 0}^{1\leq i+j\leq n}\left\{\iint_S [(i+1)x^{i+1}y^j X-(j+1)x^i y^{j+1}Y]dS\right\}^2>0.$$

(ii) If such pair of functions uniquely exist for some non-negative integer n, it is also true for all integers $m>n$. (iii) Such pair of functions always uniquely exist either for the region S or for a suitable slight deformation of S. (iv) Divide S into subregions $\{S_i\}$. Let Δ denote the maximal diameter of S_i's. Let $X_n(S_i)$ and $Y_n(S_i)$ denote the uniquely determined functions for S_i. We have

$$(6)\qquad \lim_{\Delta\to 0}\sum_{(S_i)}\iint_{S_i}\{[X_n(S_i)-X]^2+[Y_n(S_i)-Y]^2\}dS_i=0.$$

From the author's summary.

★**Hoheisel, Guido. Gewöhnliche Differentialgleichungen.** Sammlung Göschen, Band 920. Fünfte, durchgesehene Auflage. Walter De Gruyter & Co., Berlin, 1956. 129 pp. DM 2.40.

Langer, Rudolph E. The solutions of a class of ordinary linear differential equations of the third order in a region containing a multiple turning point. Duke Math. J. 23 (1956), 93–110.

The differential equations studied are of the form

$$(1)\qquad \frac{d^3w}{dz^3}+h_1(z, \lambda)\frac{d^2w}{dz^2}+\lambda^2\left\{h_2(z, \lambda)\frac{dw}{dz}+h_3(z, \lambda)w\right\}=0.$$

The functions h_j are powers series in λ^{-1} convergent at $\lambda=\infty$, with coefficients that are regular analytic functions of z in some domain B. This domain is assumed to contain a point z_0, where $h_2(z, \infty)$ has a simple zero. The author constructs explicit expressions which are asymptotic, as $\lambda\to\infty$, to fundamental systems of (1) in subregions of B that contain the "turning point" z_0. The relative error of the asymptotic approximations is $O(\lambda^{-m-1})$, where $m\geqq 0$ is arbitrary. To each subregion corresponds a certain sector of the λ-plane in which the asymptotic relation is valid. The set of all these subregions and sectors is such that for every z in a neighborhood of B and every of arg λ asymptotic relations can be found. Since the turning point z_0 is common to all subregions for all values of arg λ these results make it possible to calculate asymptotic expressions for the analytic continuation of any solution into the whole neighborhood of z_0 and into the whole neighborhood of $\lambda=\infty$.

The author's asymptotic expressions are written in terms of the solutions of the differential equation

$$(2)\qquad \frac{d^3v}{dx^3}+\lambda^2\times\frac{dv}{dx}+3\mu(\lambda)\lambda^2 v=0,$$

where $\mu(\lambda)$ is a certain polynomial in λ^{-1}. The asymptotic theory of (2) was developed by the author in Trans. Amer. Math. Soc. 80 (1955), 93–123 [MR 17, 365]. The results of the present paper are proved by a technique of comparison which is by now classical and whose power has been greatly enhanced in the last 25 years, largely through the work of the author himself: Starting from the known asymptotic theory of a special differential equation — in the present problem equation (2) — a "related differential equation" is constructed whose asymptotic character is also known and whose coefficients differ from those of the given differential equation by terms of order $O(\lambda^{-m-1})$. The difference of the solutions of the given and related equations is then represented as the solution of a Volterra integral equation and appraised with the help of the asymptotic properties of the related equation.

W. Wasow (Madison, Wis.).

Latyševa, K. Ya. Normal series as solutions of linear differential equations in the case of multiple roots of the characteristic equations. Ukrain. Mat. Ž. 4 (1952), 124–136. (Russian)

The nth order equation

$$y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=0$$

is considered where $P_i(x)=x^{\beta_i}(P_{i0}+P_{i1}x^{-1}+\cdots)$, where β_i are integers. First, necessary and sufficient conditions are given in the case of multiple roots of the characteristic equation for the existence of a solution asymptotic to $\exp(Q(x))x^r\Psi(x)$, where Q is a polynomial and Ψ is a series of inverse powers of x not necessarily convergent. Later formal solutions of the form

$$\exp(Q(x))\sum_1^n x^{r_i}\Psi_i(x)\ln^{i-1}x$$

are considered. *N. Levinson* (Cambridge, Mass.).

Myškis, A. A theorem from the theory of dynamical systems. Moskov. Gos. Univ. Uč. Zap. 145, Mat. 3 (1949), 129–130. (Russian)

In a topological space R let a continuous flow be given. Let S be the union of all trajectories contained in a given closed set F. Then each component of connectivity K of $R-S$ contains at least one component of $R-F$. In order that $K\cap(R-F)$ should contain exactly one component of $R-F$ it is necessary and sufficient that each trajectory intersect at most one component of $R-F$. For example, if u is a function of class C' in a domain R of n-space, if the trajectories of the system $dx_i/dt=\partial u/\partial x_i$ $(i=1,\cdots,n)$ lie entirely in R, if $u=0$ on the boundary of a compact set $F\subset R$, and if any two components of $R-F$ are separated by a positive distance, then the union S of the trajectories contained in F separates all the components of $R-F$ from each other. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Amerio, Luigi. Soluzioni quasi-periodiche, o limitate, di sistemi differenziali non lineari quasi-periodici, o limitati. Ann. Mat. Pura Appl. (4) 39 (1955), 97–119.

1) A general criterion for the existence of almost periodic (a.p.) solutions of system $\dot{x}=f(t,x)$ is given; this includes criteria of J. Favard [Acta Math. 51 (1927), 31–81] for linear systems. Only a special case will be quoted here: let f be continuous in t and x, a.p. in t (uniformly in x when x is confined to any bounded set), and let $f^*(t,x)=\lim_{n\to\infty}f(t+h_n,x)$, the limit being uniform in t and uniform in x on bounded sets. If, for each such f^*, the system $\dot{x}=f^*(t,x)$ has only one solution bounded for all t, then this solution will be a.p.

2) For the system $\ddot{x}+\phi(\dot{x})+Ax=f(t)$, suppose that (i) the matrix A is positive definite and symmetric, (ii) f is continuous and $|f(t)|\leqq M$, (iii) ϕ is continuous,

$$\liminf\frac{\phi(y)\cdot y}{|\phi(y)|\,|y|}=h>0,\ \liminf|\phi(y)|>\frac{M}{h}.$$

(lim inf taken for $|y|\to\infty$, dot denotes scalar product). Then: each solution is bounded as $t\to+\infty$; there exists a solution bounded for all t; if $f(t)$ is periodic, there exists a periodic solution with the same period.

3) For the same system as in 2), add to (i)–(iii) the further condition (iv) $(\phi(u+v)-\phi(u))\cdot v>0$ (all u, all $v\neq0$). Then the bounded solution is unique, and all other solutions converge to it as $t\to+\infty$. From this result, and the result stated in 1), it follows that if $f(t)$ is a.p. then there exists a unique a.p. solution. *G. E. H. Reuter.*

Vrkoč, Ivo. On the inverse theorem of Četaev. Czechoslovak Math. J. 5(80) (1955), 451–461. (Russian. English summary)

Let $\dot{x}=X(t,x)$ where $X(t,x)$ is of class C^1 on $\Omega(r):t\geqq0$, $\|x\|<r$ and $X(t,0)\equiv0$ on $t\geqq0$. The author proves that if $x=0$ is unstable in Lyapunov's sense there exists on some $\Omega(\rho)$, $\rho<r$, a function $V(t,x)$ which is of class C^1, non-negative and bounded, such that $V=\partial V/\partial t+\operatorname{grad}V\cdot X$ on $\Omega(\rho)$ and, given any $\varepsilon>0$, there is an x in $\|x\|<\varepsilon$ for which $V(0,x)>0$. This result implies that the conditions of N. G. Četaev [Doklady Akad. Nauk SSSR (N.S.) 1 (1934), 529–531] and Lyapunov [Problème général de la stabilité du mouvement, Princeton, 1947, p. 265; MR 9, 34] for the instability of $x=0$ are not only sufficient but necessary as well. *H. A. Antosiewicz* (Washington, D.C.).

Krasovskiĭ, N. N. On the inversion of theorems in the second method of A. M. Lyapunov for investigation of stability of motion. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 3(69), 159–164. (Russian)

Consider $\dot{x}=X(x)$ where X is of class C^m on some region R in Euclidean n-space and denote by $f(p,t)$ the trajectory passing through p at $t=0$. Let $H(r,p)$ be the intersection of the sphere $S(r,p)$ with the hyperplane through p normal to $f(p,t)$. The region $G(2\varepsilon,p)=f(H(2\varepsilon,p),t)$, $t\in(-\infty,\infty)$ is called a tube if a trajectory in G intersects H only at $t=0$ and if the boundary of G consists of points $f(q,t)$ with $q\in bd\cdot H$. Relative to a tube $G(2\varepsilon,p)$, the author proves the existence of a function $V_p(x)$ of class C^m on R such that $\operatorname{grad}V_p\cdot X>0$ inside and $V_p\equiv0$ outside $G(\varepsilon,p)$, and $V_p=0$ on $H(2\varepsilon,p)$. Using this result, he then proves the converse of Lyapunov's theorem on instability [Problème général de la stabilité du mouvement, Princeton, 1947, p. 265; MR 9, 34]: If the solution $x=0$ of $\dot{x}=X(x,t)$ where X is of class C^1 on $\|x\|<r$, $t\geqq0$ is unstable, there exists a function $V(x,t)$ of class C^1, $V\to0$ with x uniformly in t, such that every neighborhood of $x=0$ contains points for which $V>0$, whatever $t>0$, and such that $\partial V/\partial t+\operatorname{grad}V\cdot X=\lambda V+W$, where $\lambda>0$ and $W(x,t)\geqq0$. A different proof of essentially the same result is due to Vrkoč [see the paper reviewed above]. The author also gives a new proof, for autonomous equations, of the converse of Lyapunov's theorem on asymptotic stability [cf. Massera, Ann. of Math. (2) 50 (1949), 705–721; 64 (1956), 182–206; MR 11, 721; 17, 42; Barbašin, Mat. Sb. N.S. 29(71) (1951), 233–280; MR 13, 756]. *H. A. Antosiewicz.*

Gu, Čao-hao. On the stability of two equations. Acta Math. Sinica 4 (1954), 347–357. (Chinese. Russian summary)

The present note considers the problem of stability for the system of equations

$$\frac{dx}{dt}=f_1(x)+ay,\quad\frac{dy}{dt}=f_2(x)+g(x)y.$$

From the author's summary.

Četaev, N. G. On some problems of stability of motion in mechanics. Prikl. Mat. Meh. 20 (1956), 309–314. (Russian)

This is an expository discussion of some problems in mechanics that are concerned with the stability of motion. *H. A. Antosiewicz* (Washington, D.C.).

Krasovskiĭ, N. N. On the application of the second method of Lyapunov for equations with time retardations. Prikl. Mat. Meh. 20 (1956), 315–327. (Russian)

Lyapunov's second method is extended to systems

$$\dot{x}_i(t)=X_i[x_1(t-h_{i1}(t)),\cdots,x_n(t-h_{in}(t)),t],\quad 1\leqq i\leqq n,$$

where $X(x,t)$ is lipschitzian on $\|x\|\leqq H$, $t\geqq0$ and $X(0,t)\equiv0$ for $t\geqq0$, and where the functions $h_{ij}(t)$, $1\leqq i,j\leqq n$, are piecewise continuous and satisfy $0\leqq h_{ij}(t)\leqq h_j=\text{const}$ for

$f \geqq 0$. After suitable modification of the basic stability definitions, the author proves sufficient conditions for the asymptotic and total stability of $x=0$, necessary and sufficient conditions for the uniform asymptotic stability and theorems relating exponential-asymptotic stability to total stability. Their precise statements are too long to be stated more explicitly here. All results are based upon considerations of positive definite functions in the manner of Lyapunov's second method. *H. A. Antosiewicz.*

Volpato, Mario. Rettifica alla memoria: Sopra un problema di valori al contorno per l'equazione differenziale $y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}, \lambda)$. Rend. Sem. Mat. Univ. Padova **25** (1956), 273–278.

The author states that an argument in the earlier paper [same Rend. **23** (1954), 224–244; MR **15**, 709] is now seen to depend on continuous convergence, supplies the needed assumption, and indicates the resulting modification in the proof. *F. A. Ficken* (Knoxville, Tenn.).

Friedman, Bernard; and Mishoe, L. I. Eigenfunction expansions associated with a non-self-adjoint differential equation. Pacific J. Math. **6** (1956), 249–270.

This paper deals with the problem of the expansion of an arbitrary function $f(x)$ in eigenfunctions of the ordinary differential equation

$$u''+q(x)u+\lambda(p(x)u-u')=0$$

with the end-point conditions $u(0)=u(1)=0$. A related class of expansions for non-self adjoint second-order ordinary differential operators was discussed by R. E. Langer [Trans. Amer. Math. Soc. **31** (1929), 868–906]. An expansion in eigenfunctions of $A+\lambda B$ for A a non-self-adjoint elliptic differential operator, B positive with Dirichlet boundary conditions on an n-dimensional domain was given by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 433–439; MR **14**, 984]. In the present case, however, B is not positive while, since one has ordinary differential operators, much more precise information can be obtained about the asymptotic structure of the eigenfunctions by essentially using the method of variation of constants. Applying such estimates the authors obtain the following result: Let p and q be real, q continuous and p with a continuous second derivative on the interval $[0, 1]$. Let F be of bounded variation on the interval. Then there exists a discrete set of eigenvalues λ_n of the given problem, a corresponding sequence of eigenfunctions $u_n(x)$, and a conjugate sequence of eigenfunctions of the adjoint equation $v_n(x)$, normalized by the conditions $\int_0^1 u_j(x)(p(x)v_k+v')dx=\delta_{jk}$. If one forms the series $\sum a_n u_n(x)$, where

$$a_n=\int_0^1 F(t)(pv_n+v_n')dt,$$

then this series converges to

$$g(x)=\tfrac{1}{2}[F(x+0)+F(x-0)-c\exp(\int_0^x (p(t)dt)],$$

where $c=\frac{1}{2}[F(0+)+F(1-)\exp(-\int_0^1 p(t)dt)]$. In particular if $c=0$, the series converges to $F(x)$ at every point of continuity. *F. Browder* (New Haven, Conn.).

Douglas, A. S. On the Sturm-Liouville equation with two-point boundary conditions. Proc. Cambridge Philos. Soc. **52** (1956), 636–639.

Let $P(x, \lambda)$ be a family of solutions of

(1) $$P''(x)+[g(x)+\lambda f(x)]P(x)=0$$

subject to $P(x_0, \lambda)=0$ and differentiable with respect to the parameter λ. From (1) the author derives the relation

(2) $$P^2(X)\Big[\frac{\partial}{\partial\lambda}(P'/P)\Big]_{x=X}=\int_{x_0}^{X} fP^2\,dx.$$

He uses this to estimate the error made in approximating an eigenfunction $P(x, \bar{\lambda})$ by means of $P(x, \lambda)$, where λ is an approximation with known error estimate to the unknown eigenvalue $\bar{\lambda}$.

The formula (2) is also used as the basis of an iterative scheme for finding eigenvalues. For example if $\bar{\lambda}$ is to be the eigenvalue of (1) with the end conditions $P(x_0)=P(x_n)=0$, one defines $Q(x, \lambda)$ and $R(x, \lambda)$ as solutions of (1) satisfying $Q(x_0)=0$ and $R(x_n)=0$. $\bar{\lambda}$ is characterized by the fact that Q'/Q and R'/R match at some intermediate point X. This condition is approximated by a Newton's method based on (2). *H. Weinberger.*

Šidlovskaya, N. A. Application of the method of differentiation with respect to a parameter in solving nonlinear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 213–216. (Russian)

Let P_λ denote the problem of solving E_λ: $y''+\lambda f(x, y, y')=0$ for $a\leq x\leq b$ subject to $y(a)=A$, $y(b)=B$. A method of solving P_1 is first to solve P_0, getting $y_0(x)$, and then to continue the solution $y(x, \lambda)$ of P_λ, where $y(x, 0)=y_0(x)$, over the interval $0\leq\lambda\leq 1$, thus obtaining a solution $y(x, 1)$ of P_1. It is observed that y_λ $(=\partial y/\partial\lambda)$ satisfies the linear ordinary differential equation.

(L) $$y_\lambda''+\lambda f_{y'}y_\lambda'+\lambda f_y y_\lambda+f=0$$

and vanishes at $x=a$ and $x=b$. If $Q(x, \lambda; y)$ solves this linear problem, then $y(x, \lambda)$ solves the problem $dy/d\lambda=Q(x, \lambda; y)$, $y(x, 0)=y_0(x)$. Thus P_1 has been reduced to a linear second order problem and a nonlinear first order problem. A numerical process is indicated for getting in n equal steps from $\lambda=0$ to $\lambda=1$ using (L) without using the nonlinear relation $y_\lambda=Q$. A similar formal methods is displayed for a Dirichlet problem. *F. A. Ficken.*

Jones, John, Jr. On some third order non-linear differential equations. Portugal. Math. **14** (1956), 95–98.

The author remarks how the boundedness of some rather restrictive particular solutions of certain nonlinear differential equations of third order can be obtained from known results of nonlinear differential equations of second order by change of variables. *C.-T. Taam.*

Gel'fand, I. M. On identities for eigenvalues of a differential operator of second order. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 191–198. (Russian)

The author considers the characteristic values λ_n associated with $-y''+p(x)=\lambda y$, $y(0)=y(\pi)=0$. The asymptotic relation $\lambda_n=n^2+c_0+c_2n^{-2}+\cdots$ is recalled. This relation is inverted to give $n^2=\lambda+b_0+b_1\lambda^{-1}+\cdots=\Psi(\lambda)$. The asymptotic relation for large ζ

$$\sum_1^\infty (\lambda_n+\zeta)^{-1}=\sum_0^\infty (-1)^k A_k\zeta^{-k-1}+\frac{d}{d\zeta}(-\Psi(-\zeta))^{\frac{1}{2}}$$

is proved.

$$A_1=\sum_1^\infty (\lambda_n-n^2-c_0)-\tfrac{1}{2}c_0,$$

$$A_2=\sum_1^\infty (\lambda_n{}^2-n^4-2n^2c_0-2c_1-c_0{}^2)-\tfrac{1}{2}(2c_2{}^2+c_0{}^2),\ \text{etc.}$$

Other results are given. *N. Levinson.*

See also: Kreyszig, p. 130; Onoyama, p. 156; Požarickiĭ, p. 161; Birkhoff, p. 168.

Partial Differential Equations

Sagan, H. Über ein, einer selbstadjungierten Differentialgleichung zuordenbares dreidimensionales Variationsproblem. Österreich. Ing.-Arch. **10** (1956), 264–267.

The author makes the rather obvious remark that if a functional is rendered stationary by a function which is a product of functions of single variables, then this solution also renders the functional stationary among the class of such product functions. *H. Weinberger.*

Lions, Jacques Louis. Sur quelques problèmes aux limites. C. R. Acad. Sci. Paris **242** (1956), 3028–3030.

Etude du problème aux limites de Dirichlet pour des opérateurs $A+L$ avec

$$A=\sum_{p,q}(-1)^{|p|}D_x^p(a_{pq}(x,t)D_x^q),\ |p|,|q|\leqq m,$$

opérateur elliptique en un sens précisé et L opérateur différentiel convenable à coefficients indéfiniment dérivables. L'auteur pose un problème de Dirichlet généralisé dans un ouvert Ω non nécessairement cylindrique de l'espace-temps, la donnée étant un second membre adéquat, et la condition de Dirichlet s'exprimant par le fait que la fonction inconnue et ses dérivées par rapport à x jusqu'à l'ordre m sont limites dans $L_2(\Omega)$ des dérivées correspondantes de fonctions indéfiniment dérivables à support compact dans Ω. Il montre l'existence d'au moins une solution.

Après diverses hypothèses sur Ω, l'auteur pose un problème concernant le même opérateur dans lequel $L=\partial/\partial t$ et A est elliptique dans un sens plus large, problème qui comporte des données initiales et au second membre. Il existe cette fois une solution unique.

La note ne contient pas de démonstrations détaillées. *H. G. Garnir* (Liège).

Kreyszig, Erwin. On certain partial differential equations and their singularities. J. Rational Mech. Anal. **5** (1956), 805–820.

The present paper uses Bergman's integral operator

$$U(z,z^*)=\int_{-1}^{1}E(z,z^*,t)f[(z/2)(1-t^2)]dt/(1-t^2)^{\frac{1}{2}}$$

to relate singularities in the coefficients of the elliptic partial differential equation

$$\Delta u+a(x,y)(\partial u/\partial x)+b(x,y)(\partial u/\partial y)+c(x,y)u=0 \tag{2}$$

to the singularities of the solution u. This is a significant corollary of the author's earlier paper [same J. **4** (1955), 907–923; MR **17**, 492], in which he assumed an exponential form for the generating function $E(z,z^*,t)$ and obtained explicit relations between the coefficient of (2) (rewritten in complex notation with $z_1=x+iX$, $z_2=y+iY$, $z=z_1+iz_2$, $z^*=z_1-iz_2$) and the terms appearing in $E(z,z^*,t)$, for all equations (2) which admit a generating function of the assumed form. The extension in the present paper consists mainly in obtaining an ordinary differential equation

$$\sum_{i=0}^{k}B_i(z_1,z_2)(d^iu/dz_1^i)=0 \tag{3}$$

which is satisfied by u in the plane $z_2=z_2^{(0)}=$const, considered as a subset of the four-dimensional space of the two complex variables z_1 and z_2. The equation (3) is of Fuchsian type, so that the relation between the singularities of solutions and coefficients is known, the relation between the singularities of the coefficients of (3) and the behavior of $E(z,z^*,t)$ can be determined, and that between $E(z,z^*,t)$ and the coefficients of (2) follows from the explicit relations obtained in the earlier paper just mentioned. *R. B. Davis* (Syracuse, N.Y.).

Maurin, L. Die Lösung im Grossen gemischter Probleme für Systeme inhomogener linearer Differentialgleichungen. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 179–185.

A solution of the Cauchy problem for the differential equation $u_{tt}=Au+f(t)$ ($t>0$, u and $f(t)$ elements of a Hilbert space H, A a self-adjoint linear operator) can be obtained by the operational calculus when A is bounded from below. The author considers the case when A is an extension of an elliptic differential operator a with smooth coefficients defined in a region Ω of real n-space and H is the space of square integrable functions in Ω with values in a finite-dimensional Hilbert space and states that if $u(0)\in D(A)$ and $u_t(0)$, $f(t)\in D(\sqrt{A})$, then the solution can be obtained in terms of the Fourier expansion associated with A (it is no restriction to assume that $A\geqq 0$). Conditions for the solution to be of class C^k, (k the order of a) are given weaker than those of Ladyženskaya [The mixed problem for a hyperbolic equation, Gostehizdat, Moscow, 1953; MR **17**, 160]. It is also announced that the Dirac equation for stationary fields can be subordinated to the theory of self-adjoint operators. *L. Gårding* (Lund).

O'Sullivan, D. G. Treatment of the equations of classical diffusion in homogeneous isotropic media. J. Chem. Phys. **25** (1956), 270–274.

The author considers the diffusion of a solute in a system consisting of n adjacent regions, each homogeneous and isotropic. In the region labelled ν the concentration v, is given by the solution of the equation

$$\frac{\partial v_\nu}{\partial t}=D_\nu\nabla^2v_\nu+kv_\nu+A_\nu(r,t). \tag{1}$$

Here r is a coordinate of position, D_ν and k are constants, the latter throughout the whole system. Conditions at the two outer boundaries are included in

$$k_1\frac{\partial v}{\partial x}+k_2\frac{\partial v}{\partial y}+k_3\frac{\partial v}{\partial z}+k_4v=g(r,t),$$

where $k_1,\ \cdots,\ k_4$ are functions of position only not all zero everywhere. At the interfacial boundaries there are conditions of the type

$$K_{\nu,1}\frac{\partial v_\nu}{\partial x}+K_{\nu,2}\frac{\partial v_\nu}{\partial y}+K_{\nu,3}\frac{\partial v_\nu}{\partial z}+K_{\nu,4}v=$$
$$K_{\nu+1,1}\frac{\partial v_{\nu+1}}{\partial x}+K_{\nu+1,2}\frac{\partial v_{\nu+1}}{\partial y}+K_{\nu+1,3}\frac{\partial v_{\nu+1}}{\partial z}+K_{\nu+1,4}v_{\nu+1},$$

where again the K's are functions of position only.

Using Duhamel's theorem the solution may be deduced from that of the analogous problem, where A_ν and g are functions of position only. However the solution of that problem is as yet only known in a few simple cases. Finally, the author gives a transformation to deduce the desired solution from that of the analogous problem, where moreover equation (1) is replaced by $\nabla^2u=0$. As examples he gives a) a hollow sphere consisting of a

single region and b) an infinite cylinder consisting of a single region.

The author suggests that only trivial modifications will be required in treating heat conductivity problems.

H. Bremekamp (Delft).

Fava, Franco. Sulle varietà integrali del sistema

$$\begin{cases} x_{uv}=a_1x_u+a_2x_v+a_3x_w+ax \\ x_{uw}=Lx_{vw}+b_1x_u+b_2x_v+b_3x_w+bx. \end{cases}$$

Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 81–116.

Considerando come funzioni tuttavia indeterminate delle tre variabili u, v, w i coefficienti delle due equazioni in derivate parziali quali appaiono nel titolo, l'A. si propone anzitutto di determinare le condizioni a cui dette funzioni debbono soddisfare affinchè derivazioni successive non aggiungano al sistema proposto nuove equazioni supplementari. Una osservazione dello stesso A. in terza pagina, affermando che mediante „calcoli non brevi che non riteniamo necessario riportare" risulta dimostrata una gran limitazione circa tali condizioni, meriterebbe forse una revisione accurata per garantire il risultato accennato, il quale in seguito a un rapido tentativo circa tali calcoli ci è parso dubbioso. — L'A. passa successivamente alla traduzione geometrica delle condizioni conseguite, considerando finalmente nell'ultimo paragrafo le conseguenze di particolari condizioni che si ottengono anullando, mediante la scelta delle suddette funzioni, determinati invarianti delle equazione proposte. *B. Levi* (Rosario).

Fichera, Gaetano. Su un principio di dualità per talune formole di maggiorazione relative alle equazioni differenziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 411–418 (1956).

A corollary to the basic Theorem I of a previous paper of the author [Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, pp. 174–227; MR 17, 626] is noted and applied to elliptic, parabolic, and biharmonic equations. The corollary states that the existence of certain a priori estimates implies the existence of "dual" estimates. Application to elliptic equations: consider the elliptic operator

$$L(v)\equiv\sum_{h,k}^{1,r}a_{hk}\frac{\partial^2 v}{\partial x_h\partial x_k}+\sum_{h=1}^{r}b_h\frac{\partial v}{\partial x_h}+cv,$$

where the a_{hk} are of class 2, the b_h of class 1 and c is a positive function satisfying a Hölder condition, all in domain D with boundary $\mathfrak{F}D$. Let v, of class 1 in D, satisfy $L(v)=0$ and p be an even number. Then from the inequality

$$\Big(\int_D v^p dx\Big)^{1/p}\leq K_p\Big(\int_{\mathfrak{F}D}v^p d\sigma\Big)^{1/p}$$

follows the "dual" inequality

$$\text{(d)}\quad \Big(\int_{\mathfrak{F}D}\Big|\frac{\partial u}{\partial \nu}-bu\Big|^q d\sigma\Big)^{1/q}\leq K_p\Big(\int_D |L^*(u)|^q dx\Big)^{1/q},$$

where

$$\frac{1}{p}+\frac{1}{q}=1;\quad \frac{\partial u}{\partial \nu}=\sum_{h,k}^{1,r}a_k\frac{\partial u}{\partial x_k}\cos(x_k,N);$$

N is the inward directed normal on $\mathfrak{F}D$;

$$b=\sum_{h=1}^{r}\Big(b-\sum_{k=1}^{r}\frac{\partial a_{hk}}{\partial x_k}\Big)\cos(x_h,N);$$

$$L^*(u)=\sum_{h,k}^{1,r}\frac{\partial^2}{\partial x_h\partial x_k}(a_{hk}u)-\sum_{h=1}^{r}\frac{\partial}{\partial x_h}(b_hu)+cu;$$

and u is an arbitrary function of class 1 in D, class 2 in $D-\mathfrak{F}D$, such that $u=0$ on $\mathfrak{F}D$ and $|L^*(u)|^q$ is summable in D. {Since $u=0$ on $\mathfrak{F}D$, the term bu in (d) may be omitted. On p. 413, read $L^*(w)>0$ on $D+\mathfrak{F}D$ for $L^*(w)>0$ on D; on p. 414, read L^* for E^*; on p. 415, read D for A}.

Jane Cronin (New York, N.Y.).

Cimmino, Gianfranco. Sulla nozione di sistema differenziale aggiunto per i problemi ai limiti lineari in più variabili. Ann. Mat. Pura Appl. (4) **40** (1955), 223–238.

Let D be a closed bounded domain of the x, y-plane whose boundary has piecewise continuous curvature. Consider the differential equation

$$(1)\qquad \Delta^m u+p_1\Delta^{m-1}u+\cdots+p_{m-1}\Delta u+p_mu=0,$$

where Δ is Laplace's operator and the p_{m-1} are bounded measurable functions in D. Unless the p_r have sufficiently many derivatives (1) does not possess an adjoint differential equation in the usual sense. Following ideas developed by various mathematicians including himself the author constructs an integral equation which can be regarded as the adjoint problem of (1), because its solutions satisfy the differential equation adjoint to (1) in the elementary sense, whenever the p_r are sufficient regular. In order to arrive at this integral equation the author considers the functions f which satisfy for all polynomials ω the orthogonality relation

$$(2)\qquad \iint_D f\cdot(\Delta^m\omega+p_1\Delta^{m-1}\omega+\cdots+p_{m-1}\Delta\omega)\,dxdy=0.$$

These functions are, according to the usual definition, weak solutions of the adjoint problem, if it exists. By a proper choice of a family of polynomials ω and subsequent passages to the limit an integral equation is obtained which is satisfied almost everywhere by f. It then follows that there exists a continuous function v that coincides almost everywhere with f and satisfies the integral equation everywhere. This integral equation is of the form

$$(3)\quad v(x_0,y_0)=\frac{1}{2\pi r}\int_{FC}v\cdot(1+\sum_{h=1}^{m-1}a_hr^{2h})ds+\frac{1}{\pi}\iint_C v\cdot(\sum_{i=1}^{m}p_iH_i-\sum_{h=1}^{m-1}ha_h\varrho^{2(h-1)})dxdy.$$

Here C is a circle in D about x_0, y_0 of arbitrary radius r, FC its boundary and the a_h and H_i are certain functions of $\varrho^2=(x-x_0)^2+(y-y_0)^2$ that can be calculated explicitly. (3) may be regarded as a generalization of the mean value theorem for harmonic functions.

This method can be extended to functions that satisfy, in addition, boundary conditions of the form

$$\sum_{j=1}^{m}\Big(\alpha_{ij}\frac{\partial\Delta^{m-j}u}{\partial n}+\beta_{ij}\Delta^{m-j}u=0\ (i=1,2,\cdots,m),$$

by subjecting the orthogonality condition that defines the weak solutions of the adjoint boundary problem in the elementary sense (if the latter exists) to similar transformations.

The arguments extend also to the case that boundary conditions are prescribed on more than one contour inside D and satisfies there certain jump conditions.

Finally, the case of a system of differential equations that contains (1) as a special case is treated in a similar fashion.

W. Wasow (Cambridge, Mass.).

Owens, O. G. Polynomial solutions of the cylindrical wave equation. Duke Math. J. **23** (1956), 371–383.

The conical characteristic value problem for the cylindrical wave equation is the determination in the interior of the characteristic cone $t^2=x^2+y^2$ ($t>0$) of that solution $u(x, y, t)$ which takes prescribes values $\psi(x, y)$ on the cone.

If a solution exists, then, by Asgeirsson's mean-value theorem,

$$-u(0, 0, T)=\psi(0, 0)+\frac{\sqrt{T}}{2\pi}\iint\frac{\{x\psi_x+y\psi_y\}}{r^2\sqrt{(T-2r)}}\,dxdy,$$

where $r^2=x^2+y^2$, and integration is over $0\leqq 2r\leqq T$. By using a Lorentz transformation which leaves invariant the cylindrical wave equation and the cone $t^2=x^2+y^2$ and transforms the point (x_0, y_0, t_0) into $(0, 0, T)$ where $T^2=t_0^2-x_0^2-y_0^2$, this formula gives the solution at any interior point (x_0, y_0, t_0) of the characteristic cone.

If $\psi(x, y)\in C^4$, this method does, in fact, give the required solution, and all second-order partial derivatives of u exist and are uniformly bounded on the closure of the interior of the characteristic cone.

As a direct proof of these facts would be difficult, the author proves first that the characteristic value problem has a unique polynomial solution which takes prescribed polynomial values on the cone. The proof is then completed by using the Weierstrass approximation theorem, so that $\psi(x, y)$ and its derivatives can be uniformly approximated by a sequence of polynomials and their corresponding derivatives. *E. T. Copson.*

Mackie, A. G. The generalized radially symmetric wave equation. Proc. Roy. Soc. London. Ser. A. **236** (1956), 265–277.

In this paper a detailed study is made of the solutions of

$$\frac{\partial^2\phi}{\partial R^2}+\frac{k}{R}\frac{\partial\phi}{\partial R}-\frac{\partial^2\phi}{\partial T^2}=0$$

in the region $R\geqq 0$, $T\geqq 0$, with Cauchy data on $T=0$. Use is made of contour integral solutions of an equation occurring in gas dynamics previously discussed by the author. [Proc. Cambridge Philos. Soc. **50** (1954), 131–138; **15**, 365]. The solution is obtained first for values of $T\leqq R$, but is continued into the whole quarter plane. This continuation is unique, but is not in general the analytical continuation even for analytic Cauchy data.

Different interpretations of the solution are examined. For instance, if k is a positive integer, the equation is that satisfied by radially symmetric solutions of the equation of wave motions in space of $k+1$ dimensions, and the solution of the initial value problem for this latter equation is deduced. *E. T. Copson* (St. Andrews).

Poritsky, H. Expression of wave functions over a half space in terms of their boundary values. Quart. Appl. Math. **14** (1956), 185–197.

It follows from Kirchhoff's formula that a solution u of the wave equation can be expressed in the half-space $z>0$ in terms of boundary values on the plane $z=0$ by either of the formulae

$$u(x, y, z, t)=\frac{1}{2\pi}\int\left\{[u]+\frac{R}{c}\left[\frac{\partial u}{\partial t}\right]\right\}d\omega,$$

$$u(x, y, z, t)=-\frac{1}{2\pi}\int\left[\frac{\partial u}{\partial z}\right]\frac{dS}{R},$$

where integration is over the plane $z=0$. Square brackets denote retarded values, R is the distance from (x, y, z) to the integration-point and $d\omega$ is the solid angle subtended by the element of area dS at (x, y, z). These results are used to deduce from known wave functions the values of certain definite integrals. Thereafter the extension of the results to an arbitrary number of spatial dimensions is considered. *E. T. Copson* (St. Andrews).

Chiffi, Antonio. Analisi esistenziale e quantitativa dei problemi di propagazione. Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 247–281 (1956).

In a Euclidean space S_{m+1} of $m+1$ dimensions, consider a semi-indefinite cylinder S formed of points $Q\equiv(P, t)$ where $t\geqq 0$ and P belongs to a bounded domain D of a space S_m, in which position is fixed by coordinates $(x_1, x_2, \cdots, x_m)$. Then the object is to find a function $u(P, t)$, defined in S, which, together with its first and second derivatives, satisfies certain conditions of regularity and which is such that

$$-\sum_{i,k}\frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial u}{\partial x_i}\right)+a\frac{\partial^2 u}{\partial t^2}+\sum_k 2a_k\frac{\partial^2 u}{\partial x_k\partial t}+b\frac{\partial u}{\partial t}+\sum_k b_k\frac{\partial u}{\partial x_k}+cu=f,$$

where a, a_i, b, b_i, c, f are defined in S, $a>0$, but a_{ik} is a function only of the x_k; moreover, the operator

$$\sum_{i,k}\frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial u}{\partial x_i}\right)$$

is of elliptic type. This solution has also to satisfy initial conditions $u(P, 0)=g(P)$, $u_t(P, 0)=g_1(P)$, and a boundary condition $u(P, t)=q(P, t)$ when P is on the frontier of D.

The paper is concerned with the existence, uniqueness and the approximate calculation of the solution by a practically attainable procedure.

It is impossible to give a brief account of the methods used, which are based on those developed by Stampacchia [Giorn. Mat. Battaglini (4) **3(79)** (1950), 169–208; Ricerche Mat. **1** (1952), 27–54; MR **13**, 955; **14**, 30. *E. T. Copson* (St. Andrews).

Alexiewicz, A.; and Orlicz, W. Some remarks on the existence and uniqueness of solutions of the hyperbolic equation

$$\frac{\partial^2 z}{\partial x\partial y}=f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right).$$

Studia Math. **15** (1956), 201–215.

The authors prove the following theorems: (1) let the function $f(x, y, z, p, q)$ be continuous and bounded in Q_∞:

$$|f(x, y, z, p, q)|\leqq M,$$

and let it satisfy a Lipschitz condition

$$|f(x, y, z, p_1, q_1)-f(x, y, z, p_2, q_2)|\leqq L_k((p_1-p_2)+(q_1+q_2)),$$

in every set Q_k. Then there exists a function $z(x, y)$ in R having a continuous derivative $\partial^2 z/\partial x\partial y$ and satisfying the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}=f\left(x, y, z\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

with the initial conditions $z(x, c)=\sigma(x)$, $z(z, y)=\tau(y)$. (2) Let the function $f(x, y, z)$ defined for $a\leqq x\leqq b$, $c\leqq y\leqq d$ be: 1° bounded, $|f(x, y, z)|\leqq M$; 2° measurable in (x, y) for fixed z in a set dense in $(-\infty, \infty)$; 3° continuous in z for fixed x, y. Then there exists a continuous function $z(x, y)$

satisfying the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}=f(x, y, z)$$

almost everywhere, and such that $z(x, c)=\sigma(x)$, $z(a, y)=\tau(y)$. (3) Let $f(x, y, z, p, q)$ satisfy the hypotheses of (1) and let $\|f_n-f\|\to 0$. Let $z_n(x, y)$ be a solution in R of the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}=f_n\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

with initial values $z(x, \eta_n)=\sigma_n(x)$, $z(\xi_n, y)=\tau_n(y)$. Let the solution $z(x, y)$ of the equation

$$\frac{\partial^2 z}{\partial x\,\partial y}=f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

with initial condition $z(x, \eta)=\sigma(x)$, $z(\xi, y)=\tau(y)$ be unique. If $\delta_n=(\xi_n, \eta_n, \sigma_n, \tau_n)\to(\xi, \eta, \sigma, \tau)$, then $\|z_n-z\|\to 0$. (4) The set $\mathfrak{S}$ of those $(\xi, \eta, \sigma, \tau, f)\in \mathfrak{A}\times\mathfrak{D}$ for which the equation

(*) $z(x, y)=$

$$z_0(x, y)+\int_a^x\int_c^y f\left(u, v, z(u, v), \frac{\partial z(u, v)}{\partial x}, \frac{\partial z(u, v)}{\partial y}\right)du\,dv,$$

$$z(x, c)=\sigma(x),\ z(a, y)=\tau(y)$$

has at least two different solutions in R is of Baire's first category in the space $\mathfrak{A}\times\mathfrak{D}$. (5) Let the function $f(x, y, z, p, q)$ satisfy the hypotheses of (1) and let the equation (*) have a unique solution. Then the following statements are equivalent: (a) the sequence $z_n(x, y)$ converges in R; (b) the functions $z_n(x, y)$ and their partial derivatives of the first order converge uniformly in R; (c) $|z_n(x, y)-z_{n+1}(x, y)|\rightrightarrows 0$ uniformly in R; (d) the functions $z_n(x, y)$ converge in R to a solution of (*). R: $a\leqq x\leqq b$, $c\leqq y\leqq d$. By $\mathfrak{A}$ is denoted the space of quadruples $(\xi, \eta, \sigma, \tau)=\delta$ where $(\xi, \eta)\in R$; $\tau=\tau(y)$ and $\sigma=\sigma(x)$ are two functions with continuous derivatives of first order, defined respectively for $a\leqq x\leqq b$, $c\leqq y\leqq d$ and such that $\sigma(\xi)=\eta(\tau)$. $\mathfrak{D}$ denotes the space of continuous and bounded functions $f(x, y, z, p, q)$ in Q_∞, satisfying the conditions

$$|f(x_1, y_1, z_1, p, q)-f(x_2, y_2, z_2, p, q)|\leqq \omega(\max(|x_1-x_2|, |y_1-y_2|, |z_1-z_2|)),$$

where $\omega(\delta)\to 0$ as $\delta\to 0+$,

$$|f(x, y, z, p_1, q_1)-f(x, y, z, p_2, q_2)|\leqq L_k(|p_1-p_2|+|q_1-q_2|)$$

for $(x, y, z, p_1, q_1)\in Q_k$, $(x, y, z, p_2, q_2)\in Q_k$. Q_k: $a\leqq x\leqq b$, $c\leqq y\leqq d$, $|z|\leqq k$, $|q|\leqq k$, $|p|\leqq k$; Q_∞: $a\leqq x\leqq b$, $c\leqq y\leqq d$, $-\infty<z, p, q<\infty$. *M. Pinl* (Cologne).

Morawetz, Cathleen S. Note on a maximum principle and a uniqueness theorem for an elliptic-hyperbolic equation. Proc. Roy. Soc. London. Ser. A. **236** (1956), 141–144.

Dans cette courte note, l'auteur réussit à donner la démonstration complète de l'unicité d'un problème aux limites très général, envisagé initialement par Frankl, comprenant comme cas très particulier le problème classique de Tricomi. L'équation de type mixte considérée est de la forme $K(y)u_{xx}+u_{yy}=0$; ($K(0)=0$, $yK(y)>0$). C'est celle que l'on rencontre dans les études théoriques sur les écoulements transoniques. La méthode, très élégante, consiste à établir d'abord un principe du maximum pour la fonction ψ définie par $d\psi=-2u_xu_y\,dx+(Ku_x^2-u_y^2)dy$. Les conditions d'application sont très larges; la seule condition imposée à $K(y)$ est d'avoir une dérivée première bornée; u et ψ doivent être continus dans le domaine, frontière comprise; le long de la partie du contour située dans le demi plan elliptique, l'angle de la tangente avec l'axe des x doit rester compris entre 0 et 2π.

P. Germain (Paris).

Germain, Paul. An expression for Green's function for a particular Tricomi problem. Quart. Appl. Math. **14** (1956), 113–124.

Tricomi's equation

$$T(u)=zu_{xx}+u_{zz}=0$$

is the simplest equation of mixed type, elliptic when $z>0$, hyperbolic when $z<0$. The mixed character of the equation is involved in the definition of the Tricomi problem consisting of solving the equation $T(u)=0$ in a domain Δ bounded by two concurrent characteristics AC and BC drawn in $z<0$ and an arc AMB drawn in the half plane $z>0$ when values of u are known along AMB and AC. The existence of a solution for such a problem has been proved by F. Tricomi [Atti Accad. Naz. Linci. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14** (1924), 133–247]. The author and R. Bader have considered in particular the special case for which AMB is a "normal" curve, i.e. an arc defined by $(x-x_0)^2+y^2=R^2$, $zy=2z^{3/2}$ with x_0 and R given constants [O.N.E.R.A. Publ. no. **54** (1952); Rend. Circ. Mat. Palermo (2) **2** (1953), 53–70; MR **14**, 654; **15**, 876]. In such a case it is possible to give an explicit solution of the Tricomi problem. The developments of the paper under consideration are the results of two remarks. First it was shown in Germain and Bader, Bull. Soc. Math. France **81** (1953), 145–174 [MR **15**, 432] and Germain, Comm. Pure Appl. Math. **7** (1954), 117–143 [MR **16**, 485], how it is possible to build the Green's function of a strip for a class of differential equations, even if the strip lies in a mixed region. Second, for the Tricomi equation $T(u)=0$ new independent variables can be introduced in such a way that the Green's function of the Tricomi problem is transformed into the Green's function of a strip in a mixed domain. The author introduces the following definitions

$3y=2z^{3/2}$, $r^2=x^2+y^2$, $x=rl$ ($r>0$ in Δ_0; $x>0$, $z>0$).

With $\xi=\lg r$ and $\lambda=\int_l^1(1-v^2)^{-2/3}dv$ as new independent variables the operator $T(u)$ becomes

$$T(u)=x^{-2}z(1-l^2)^{-1/3}\{u_{\lambda\lambda}+(1-l^2)^{1/3}(u_{\xi\xi}+\tfrac{1}{3}u_\xi)\}.$$

The Green's function is a fundamental solution e_P, satisfying $T(u)=\delta_{x_0, z_0}$, where δ_{x_0, z_0} is the Dirac distribution at the point $P(x_0, z_0)$. With ξ and λ, e_P is found to satisfy

$$(*)\qquad N(u)=\left(\frac{2}{3}\right)^{1/3}r_0^{-1/3}\delta_{\xi_0, \lambda_0},$$

$\delta_{\xi_0, \lambda_0}$ is the Dirac distribution for the variables ξ, λ. The author introduces the Fourier transform $U(\alpha, \lambda)$ of $u(\xi, \lambda)$. The Fourier transform of (*) is

$$(**)\ n(U)=U_{\lambda\lambda}+(1-l^2)^{1/3}\left[\frac{2}{3}i\pi\alpha-4\pi^2\alpha^2\right]U=$$

$$\left(\frac{2}{3}\right)^{1/3}r_0^{-1/3}\exp(2i\pi\alpha\xi)\delta_{\lambda_0},$$

where δ_{λ_0} is the Dirac distribution of one variable λ at the point $\lambda=\lambda_0$. To find an expression for the Tricomi problem the domain under consideration in the ξ, l-plane is the half plane $l>0$ which corresponds to the strip $\lambda_1>\lambda>\lambda_2$ (λ_2 negative) in the ξ, λ plane. One must form a solution

of (**) with convenient boundary conditions. According to the general theory there are two possibilities. These two solutions which correspond to the two Tricomi problems are (defined in the domain Δ_0) (1) a function $\mathfrak{H}_1(\lambda, \alpha)$, a solution of $n(u)=0$ which is real for $\beta=i\beta'$ (β' positive), and which tends uniformly toward zero for any $\lambda_2<\lambda<\lambda_1$, when β' tends towards $+\infty$; (2) a function $\mathfrak{H}_2(\lambda, \alpha)$ which is simply defined by $\mathfrak{H}_2(\lambda, \beta)=\mathfrak{H}_1(\lambda, \beta e^{-i\pi})$. The author gets explicit expressions for the functions $\mathfrak{H}_1$ and $\mathfrak{H}_2$. The Fourier transforms $G_P^{(1)}$ and $G_P^{(2)}$ of the Green's functions $g_P^{(1)}$ and $g_P^{(2)}$ become:

$$G^{(j)}{}_P=\left(\frac{3}{2}\right)^{1/3}r_0^{-1/3}\exp(-2\pi i\alpha\xi_0)G_j^* \quad (j=1, 2)$$

with

$$G_j^*=\begin{cases} h_jS_1(\lambda, \alpha)\mathfrak{H}_j(\alpha_0, \alpha), & (h_j \text{ is the inverse of the Wronskian of } S_1 \text{ and } \mathfrak{H}_j), \\ h_jS_1(\lambda_0, \alpha)\mathfrak{H}_j(\lambda_0, \alpha), & (\lambda_0<\lambda<\lambda_1,\ \lambda_2<\lambda<\lambda_0). \end{cases}$$

$S_1(\lambda, \alpha)$ and $S(\lambda, \alpha)$ are the solutions of $n(U)=0$ which satisfy

$$S_1(\lambda_1, \alpha)=0,\ S(0, \alpha)=0,\ \frac{\partial}{\partial\lambda}(S_1(\lambda_1, \alpha))=1,\ \frac{\partial}{\partial\lambda}S(0, \alpha)=1.$$

Finally $g_P^{(j)}$ is given by the Mellin transform

$$g_P^{(j)}=\left(\frac{2}{3}\right)^{1/3}(2i\pi r_0^{1/3})^{-1}\int_{k-i\infty}^{k+i\infty}G_j^*(rr_0^{-1})^{s-\frac{1}{3}}ds.$$

$g_P^{(1)}$ is the Green's function for the direct Tricomi problem for P in the hyperbolic half plane. $g_P^{(2)}$ is the Green's function of the conjugate Tricomi problem in Δ_0.

M. Pinl (Cologne).

See also: Meecham, p. 135; Mechovrišvili, p. 164; Lenskiĭ, p. 165; Gordov, p. 172; Chao and Weiner, p. 172; Onoyama, p. 156; Clarke, p. 157.

Difference Equations, Functional Equations

Tanaka, Sen-ichiro. On asymptotic solutions of non-linear difference equations of the first order. II. Mem. Fac. Sci. Kyusyu Univ. Ser. A. **10** (1956), 45–83.

[For part I see same Mem. **7** (1953), 107–127; MR **15**, 712.] The author treats, in this series of papers, the question of the existence and analytic behavior of solutions of the nonlinear difference equation $y(x+1)-y(x)=f(x, y(x))$ in the neighborhood of $x=\infty$. In this paper he considers the case where $\partial^k f/\partial y^k=0$, $k=1, 2, \cdots, m-1$, $\partial^m f/\partial y^m\neq 0$, at $x=\infty$, $y=g_0$, where g_0 is a value such that $f(\infty, g_0)=0$. The results are quite detailed and cannot be summarized easily.

R. Bellman.

Chakrabarti, S. C. On higher differences. III. Rend. Sem. Mat. Univ. Padova **25** (1956), 105–121.

This paper is an extension and generalization of parts I and II [same Rend. **23** (1954), 255–269, 270–276; MR **16**, 594]. The operator Δ^n as usually defined is generalized further than in papers I and II. A study of the operators thus defined is made. The Lagrange interpolation formula is given in terms of the generalized operators. Other problems studied include numerical differentiation.

T. Fort (Columbia, S.C.).

See also: Motzkin, p. 152; Theodorescu, p. 157; Šerman, p. 163.

Integral and Integro-differential equations

Gahov, F. D.; and Čerskiĭ, Yu. I. Special integral equations of convolution type and an area problem of the type of Riemann's problem. Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8. 21–33. (Russian)

The contents of this paper represent a special case of the analytical results presented in the paper reviewed below. Using the notations of the latter review, the present paper solves the homogeneous equations (1) and (2) for Case I.a. and II.b. respectively.

J. F. Heyda.

Gahov, F. D.; and Čerskiĭ, Yu. I. Singular integral equations of convolution type. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 33–52. (Russian)

In this paper the authors solve the equation

$$(1)\quad f(x)+(2\pi)^{-1}\int_0^\infty k_1(x-t)f(t)dt+(2\pi)^{-1}\int_0^\infty k_2(x-t)f(t)dt=g(x)\ (-\infty<x<\infty),$$

and also the dual equations

$$(2)\quad \begin{aligned} &f(x)+(2\pi)^{-1}\int_{-\infty}^\infty k_1(x-t)f(t)dt=g(x)\ (0<x<\infty) \\ &f(x)+(2\pi)^{-1}\int_{-\infty}^\infty k_2(x-t)f(t)dt=g(x)\ (-\infty<x<0) \end{aligned}$$

wherein, for the kernels involved, it is assumed that

$$(*)\quad k_j(x)e^{-yx}\in L(-\infty, \infty)\ (a_j\leqq y\leqq b_j;\ j=1, 2, \cdots),$$

and, for the solution function $f(x)$, that

$$(**)\quad \begin{aligned} &f(x)=f_+(x)-f_-(x), \\ &f_+(x)e^{-xy}\in L^2(-\infty, \infty)\ (y\geqq\beta), \\ &f_-(x)e^{-xy}\in L^2(-\infty, \infty)\ (y\leqq\alpha), \end{aligned}$$

where $f_+(x)$ and $f_-(x)$ are defined by $f_+(x)=\eta(x)f(x)$, $f_-(x)=-\eta(-x)f(x)$, $\eta(x)$ being the unit step function, $\eta(x)=0$ for $x<0$, $=1$ for $x>0$. Values of β and α are determined so as to insure the widest possible class of functions in (**) consistent with the existence of the integrals appearing in (1) and (2). Denoting these values by $\bar\beta$ and $\bar\alpha$, it is found that for (1)

$$(1^*)\quad \bar\beta=b_1,\ \bar\alpha=a_2,$$

and for (2)

$$(2^*)\quad \bar\beta=\min(b_1, b_2)=b,\ \bar\alpha=\max(a_1, a_2)=a.$$

For equation (1) it is then found that the widest possible class for $g(x)$, in view of (*), (**), and (1*) is

$$(***)\quad \begin{aligned} &g_+(x)e^{-yx}\in L^2(-\infty, \infty),\ y\geqq b_1 \\ &g_-(x)e^{-yx}\in L^2(-\infty, \infty),\ y\geqq a_2 \end{aligned} \Big\}\ \text{when } b_1\geqq a_2, \qquad \begin{aligned} &g_+(x)e^{-yx}\in L^2(-\infty, \infty),\ y\geqq a_2 \\ &g_-(x)e^{-yx}\in L^2(-\infty, \infty),\ y\leqq b_1 \end{aligned} \Big\}\ \text{when } b_1<a_2.$$

The following cases are then established for solving (1) and (2). For (1): Case I.a. $b_1\geqq a_2$; case I.b. $b_1<a_2$. For (2): Case II.a. $b_1<a_2$; case II.b. $b\geqq a$; case II.c. $b_2<a_1$. The cases corresponding to equality in I.a. and II.b. represent solutions obtained previously [see the paper reviewed below and I. M. Rapoport, Sb. Trud. Inst. Mat. Akad. Nauk Ukrain. SSR **12** (1949), 102–118].

The principal tool in the solution of (1) and (2) for the cases enumerated is the Riemann boundary problem for a complex contour consisting of two parallel lines. By

suitably manipulating the given equations and then taking the Fourier transform, the given equations (1) and (2) can be restated in a form corresponding directly to the boundary problem, for which the solution is given. The inequalities characterizing the cases enumerated determine the positions of the parallel lines comprising the contour and hence delimit the regions of analyticity of the transformed functions.

The solution of (1) for case I.a. turns out to be

$$(3)\quad f(x)= (2\pi)^{-1}\int_{\omega_1-\infty}^{\omega_1+\infty}\Phi^+(\zeta)e^{-ix\zeta}d\zeta-(2\pi)^{-1}\int_{ia_2-\infty}^{ia_2+\infty}\Phi^-(\zeta)e^{-ix\zeta}d\zeta,$$

where $\Phi(\zeta)$ is the solution of the Riemann boundary-value problem appropriate to this case. The function $\Phi(\zeta)$ is rather involved and will not be reproduced here.

J. F. Heyda (Indianapolis, Ind.).

Čerskiĭ, Yu. I. On some special integral equations. Kazan. Gos. Univ. Uč. Zap. **113**, no. 10 (1953), 43–55. (Russian)

Integral equations of the forms

$$f(x)+\int_{-\infty}^{\infty}a(x-t)f(t)dt+\operatorname{sgn}x=g(x),$$

$$\int_{-\infty}^{\infty}b(x-t)f(t)dt+\int_{-\infty}^{\infty}n(x,t)f(t)dt=g(x),$$

$$f(x)+\int_{-\infty}^{\infty}a(x-t)f(t)dt+\int_{-\infty}^{\infty}b(x-t)\operatorname{sgn}t\cdot f(t)dt+\int_{-\infty}^{\infty}n(x,t)f(t)dt=g(x)$$

can be converted by the Fourier transformation into singular integral equations involving Cauchy kernels. The author proves this and gives a number of detailed results on the problems which center about Noether's theorem for these equations. *D. C. Kleinecke.*

Radok, J. R. M. The integral equations of aeroelasticity. Commonwealth of Australia. Dept. of Supply. Austral. Aero. Res. Comm. Rep. ACA-**56** (1955), 32 pp.

A summary of reports **47**, **55** and **58** of the College of Aeronautics, Cranfield, England. Report **47** was reviewed in MR **13**, 184 and report **58** in MR **14**, 602.

★**Kupradse, W. D. Randwertaufgaben der Schwingungstheorie und Integralgleichungen.** Hochschulbücher für Mathematik, Band 21. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. viii+239 pp. DM 29.70.

A translation by H. G. Woschni from the original Russian work reviewed in MR **15**, 318. The German edition contains many changes, especially in the treatment of singular integral equations.

Meecham, William C. On the use of the Kirchhoff approximation for the solution of reflection problems. J. Rational Mech. Anal. **5** (1956), 323–334.

Starting from a Fredholm integral equation of the dipole layer type similar to that used for the Dirichlet problem the author derives the Kirchhoff reflection approximation as the first term in the Neumann series. The work is limited to two dimensions and the scalar case, but is easily extendable to three dimensions. *W. K. Saunders.*

Štykan, A. B. Graphical solution of Volterra integral equations. Irkutsk. Gos. Univ. Trudy. **8** (1953), no. 1, 28–35. (Russian)

The author presents a graphical treatment of the non-linear integral equation of Volterra type

$$u(x)=f(x)+\int_0^x K(x,s)\cdot\mu[u(s)]\cdot dx$$

in which the kernel $K(x, s)$ and the functions of $f(x)$ and $\mu(u)$ are given. The construction is particularly simple if the kernel has one or other of the forms

$$K(x,s)=F[\psi(x)-s] \text{ and } K(x,s)=F(\psi(x)\cdot s).$$

The graphical solution for $u(x)=1+\int_0^x e^{-zs}u^2(s)\cdot ds$ and for

$$u(x)=f(x)+\int_0^x\sqrt{(x-s)}\cdot u(s)\cdot ds$$

is worked out in detail. *W. E. Milne.*

Fubini, S. Non linear integral equation in field theory. Nuovo Cimento (10) **2** (1955), 180–182.

Low's integral equation for meson-nucleon scattering [Phys. Rev. (2) **97** (1955), 1392–1398] is re-derived by a new method. *A. Salam* (Cambridge, England).

Lyubič, Yu. I. On a class of integral equations. Mat. Sb. N.S. **38(80)** (1956), 183–202. (Russian)

The integral equations considered in this paper are of the form

$$(1)\qquad \int_0^1 a(x)f(x+t)dx=\varphi(t)\ (-\infty<t<\infty),$$

where $a(x)\in L(0, 1)$, $\varphi(t)$ is summable in every finite interval, and a solution $f(x)$ summable in every finite interval is sought. Equality in (1) is interpreted as equality almost everywhere.

Equation (1) is said to be a regular equation of rank $m\geq 0$ if $a(x)$ belongs to $C^{(m)}$, $a^{(m)}(x)$ satisfies a Lipschitz condition in $(0, 1)$, $a(1)\neq 0$, $a^{(m)}(0)\neq 0$, $a^{(k)}(0)=0$ $(0\leq k<m)$, and $\varphi(t)$ is absolutely continuous in every finite interval.

The first set of results deals with the initial value problem for (1); in this $f(x)$ is required to be equal to a given function $f^{(0)}(x)$ in an interval of length 1. The following uniqueness theorem is proved. Suppose that $a(x)\in L(0, 1)$, that the sets $\{x: a(x)\neq 0,\ 0\leq x\leq\varepsilon\}$ and $\{x: a(x)\neq 0,\ 1-\varepsilon\leq x\leq 1\}$ have positive measure for every $\varepsilon>0$, that $f(t)\in L$ in every interval strictly interior to (t_1, t_2+1), where $-\infty\leq t_1<t_2\leq\infty$, that

$$\int_0^1 a(x)f(x+t)dx=0\ (t_1<t<t_2),$$

and that $f(t)=0$ $(T<t<T+1)$, where $t_1\leq T\leq t_2$; then $f(t)=0$ in (t_1, t_2+1).

The main existence theorem for the initial value problem states that, given $f^{(0)}(x)\in L(0, 1)$ such that

$$\int_0^1 a(x)f^{(0)}(x)dx=\varphi(0),$$

there is a solution $f(x)$ of (1) such that $f(x)=f^{(0)}(x)$ in $(0, 1)$, provided that (1) is a regular equation of rank 0. The asymptotic behaviour of solutions is discussed, and conditions are given on $f^{(0)}(x)$ and $\varphi(x)$ for $f(x)$ to belong to various specified function classes. Necessary and sufficient conditions, of a rather more complicated character, are given for the existence of solutions of (1) when it is a regular equation of rank $m>0$.

Finally, if we write

$$S(\lambda)=\int_0^1 a(x)e^{\lambda x}dx,$$

$$C_f(\lambda)=\int_0^1 a(x)e^{\lambda x}dx\, f(t)e^{-\lambda t}dt,$$

$$C_k(t)=e^{-\lambda_k t}\operatorname*{Res}_{\lambda=\lambda_k}\left[\frac{C_f(\lambda)}{S(\lambda)}e^{\lambda t}\right],$$

it is shown that solutions $f(x)$ of the homogeneous equation

(2) $$\int_0^1 a(x)f(x+t)dx=0$$

can be expanded in a formal generalised Fourier series

(3) $$\sum_{k=1}^{\infty} C_k(t)e^{\lambda_k t},$$

where $C_k(t)$ is a polynomial in t; if (2) is a regular equation of positive rank, the series (3) is absolutely and uniformly convergent to $f(t)$ on any semi-infinite interval $t\geqq t_0$. An asymptotic expression for the zeros λ_k of $S(\lambda)$ is also given.

F. Smithies (Cambridge, England).

See also: Ivanov, p. 122; Langer, p. 127; Cimmino, p. 131; Masani, p. 138; Špaček, p. 157; Clarke, p. 157; Sarolea and Mayer, p. 161; Hristoforov, p. 163; Serman, p. 163; Kopzon, p. 163; Kogan, p. 169.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Fleischer, Isidore. Functional representation of partially ordered groups. Ann. of Math. (2) **64** (1956), 260–263.

The set of all real-valued continuous functions on a compact space has been characterized by the reviewer [Ann. of Math. (2) **51** (1950), 409–427; MR **11**, 525] as a partially ordered abelian group without making use of lattice concepts. In the present paper, the same problem is treated by a different method. Although lattices are still to be banished from the statements of the theorems, the author's proofs make essential use of the lattice ordered theory. Based on results of A. H. Clifford [ibid. (2) **41** (1940), 465–473; MR **2**, 4], M. H. Stone [Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 83–87; MR **2**, 318] and K. Yosida [Proc. Imp. Acad. Tokyo **17** (1941), 121–124; MR **3**, 210], the new treatment makes proofs shorter, and simplifies the five axioms of the reviewer. The original axioms I and II are replaced by the weaker requirement that there exists an archimedean element, while axiom IV is abolished altogether.

Ky Fan (Notre Dame, Ind.).

Minakshisundaram, S. On the structure of unitary symplectic groups. J. Indian Math. Soc. (N.S.) **19** (1955), 105–120 (1956).

This is a contribution to the generalized circle geometry ("symplectic geometry") of Siegel [Amer. J. Math. **65** (1943), 1–86, (quoted); MR **4**, 242] and Hua [ibid. **66** (1944), 470–488, 531–563 (not quoted); MR **6**, 124] where the complex plane is replaced by the system of all symmetric $n\times n$-matrices Z with complex elements ("Argand space" of $n(n+1)$ dimensions) Π. A generalized "Riemann sphere" Σ is introduced, whose points are represented by the hermitean $2n\times 2n$-matrices $\mathfrak{H}=\begin{pmatrix}-H & S\\ S & H\end{pmatrix}$ where H is is hermitean and S symmetric, that satisfy the condition $\mathfrak{H}^2=\mathfrak{E}$ ($2n\times 2n$ unit matrix). A point on Σ is said to be exceptional if $E-H$ is singular ($E=n\times n$ unit matrix). The formula $Z=S(E-H)^{-1}$ represents a "stereographic projection" $\Sigma\to\Pi$; to make it a one-one correspondence, all exceptional points of Σ are identified. Its inverse is given by $H=(\bar{Z}Z-E)(\bar{Z}Z+E)^{-1}$, $S=2Z(\bar{Z}Z+E)^{-1}$. Next, the author studies the group $\mathrm{USp}(2n)=\mathrm{U}(2n)\cap\mathrm{Sp}(2n)$ of all matrices $\mathfrak{M}$ that satisfy the conditions

$$\mathfrak{M}\mathfrak{F}\mathfrak{M}'=\mathfrak{F}=\begin{pmatrix}0 & E\\ -E & 0\end{pmatrix},\quad \mathfrak{M}\mathfrak{M}'=\mathfrak{E},$$

which often has not been mentioned in the literature (cf. however H. Weyl, The classical groups, p. 171). It consists of all $2n\times 2n$-matrices $\mathfrak{M}=\begin{pmatrix}A & B\\ -\bar{B} & \bar{A}\end{pmatrix}$ where $AA'+BB'=E$, $AB'=BA'$, which for real A, B are orthogonal, forming the subgroup $\mathrm{O}^{(\sigma)}\mathrm{Sp}(2n)$; this group is readily seen to be isomorphic to the group $\mathrm{U}(n)$ of all unitary matrices $U=A+iB$. The group $\mathrm{USp}(2n)$ is considered as the group of "motions" of Σ onto itself, since with $\mathfrak{H}$ also $\mathfrak{M}\mathfrak{H}\overline{\mathfrak{M}}'$ is a point of Σ, and if $\mathfrak{H}$ and $\mathfrak{M}\mathfrak{H}\mathfrak{M}^{-1}$ are points of Σ, then $\mathfrak{M}\in\mathrm{Usp}(2n)$. By means of a suitable $\mathfrak{U}=\begin{pmatrix}U & 0\\ 0 & \bar{U}\end{pmatrix}\in\mathrm{USp}(2n)$ every point $\mathfrak{H}$ of Σ can be reduced to the form $\mathfrak{U}\mathfrak{H}\bar{\mathfrak{U}}'=\begin{pmatrix}-H_d & S_d\\ S_d & H_d\end{pmatrix}$ where H_d and S_d are diagonal matrices. Then it is shown that (1) the group $\mathrm{USp}(2n)$ operates transitively on Σ, and that (2) every $\mathfrak{M}\in\mathrm{USp}(2n)$ can be written in the form $\mathfrak{U}\mathfrak{N}\mathfrak{V}$, where $\mathfrak{N}=\begin{pmatrix}A_d & B_d\\ -B_d & A_d\end{pmatrix}$ with diagonal matrices A_d, B_d and $\mathfrak{U}$, $\mathfrak{V}$ as above.

H. Schwerdtfeger.

Ramanathan, K. G. The Riemann sphere in matric spaces. J. Indian Math. Soc. (N.S.) **19** (1955), 121–125 (1956).

It is observed that the algebra of the real $n\times n$-matrices $M=A_0+A_1i+A_2j+A_3k$ whose elements are real quaternions, can be represented by the system of $2n\times 2n$-matrices

$$\begin{pmatrix}A_0+A_1i & A_2-A_3i\\ -A_2+A_3i & A_0-A_1i\end{pmatrix}$$

whose elements are complex numbers. The two systems are isomorphic. If $q=a_0+a_1i+a_2j+a_3k$, $\bar{q}=a_0-a_1i-a_2j-a_3k$, then $q^*=kqk^{-1}=a_0+a_1i+a_2j-a_3k$ and $q\to q^*$ is an involution in the system of real quaternions which can be extended to the algebra (M): Let $M=(q_{kl})$, $\tilde{M}=(\bar{q}_{lk})$; then $M^*=k\tilde{M}k^{-1}$. The matrices $U\in(M)$ that satisfy the condition $U\tilde{U}=E$ form the group $\mathrm{USp}(2n)$. The two statements (1) and (2) of the last sentence of the preceding review are proved again. The Riemann sphere is now introduced as the set of all U for which $U^*=U$, thus as the set of all fixed points of a certain involution in (M); these points are now represented (using again the otherwise entirely different notations of Minakshisundaram) by unitary matrices of the form $\begin{pmatrix}S & H\\ -\bar{H} & \bar{S}\end{pmatrix}$.

H. Schwerdtfeger (Melbourne).

See also: Rado and Reichelderfer, p. 115; Szekeres, p. 121; Gottschalk, p. 141; Berger, p. 149; Dolginov, p. 176.

Lie Groups, Lie Algebras

Harish-Chandra. On a lemma of F. Bruhat. J. Math. Pures Appl. (9) **35** (1956), 203–210.

Bruhat has given [C. R. Acad. Sci. Paris **238** (1954), 437–439; MR **15**, 504] the following lemma on a complex classical Lie group G: let H be a Cartan subgroup of G, H' the normalizer of H in G and N the nilpotent subgroup of G obtained from "negative" root vectors belonging to H. For the subgroup $U=HN$ of G, the mapping $x\to U\times U$ ($x\in H'$) then induces a one-one correspondence between the Weyl group H'/H and the set of all double cosets of G mod U. (Hence, in particular, the number of double cosets of G mod U is finite.)

The proof of Bruhat was based upon direct computation on each classical complex Lie group. In the present paper, the author studies the structure of real semi-simple Lie algebras in general and gives a new proof to the above lemma which is independent of the classification and therefore is applicable to all semi-simple Lie groups, real or complex, including exceptional groups. *K. Iwasawa.*

Topological Vector Spaces

Erdös, J. On the structure of ordered real vector spaces. Publ. Math. Debrecen **4** (1956), 334–343.

Let T be an ordered set of indices t. Let W be a family of well-ordered subsets of T containing $A+B$ with A, B. Let f be a real-valued function $f(t)$ with $D(f)=(t; f(t)\neq 0)$ in W. Then $V_T(W)$ is the ordered real vector space consisting of these functions ($f>0$ means $f(t)>0$ at the first index in $D(f)$). V_T, $V_T(F)$ are the cases in which W consists of all well ordered subsets of T, and all finite subsets of T respectively. $V_T(F)$ is a subspace of V_T.

Hausner and Wendel [Proc. Amer. Math. Soc. **3** (1952), 977–982; MR **14**, 566] gave an example of an ordered real vector space which is not isomorphic to any V_T; the present author points out that it also fails to be isomorphic to any $V_T(F)$. Hausner-Wendel's main theorem is: every real ordered vector space V_0 is isomorphic to a suitable subspace of V_T with T as the ordered set of V_0's Archimedean-equivalence classes; the present author shows that this subspace can be chosen to be $V_T(F)$ whenever V_0 has countably infinite dimension (in this case V_0 has a basis consisting of suitable vectors, one from each Archimedean-equivalence class).

An explicit construction is given for ordering an arbitrary real vector space (and in all possible ways). It is pointed out that Archimedean vector lattices of countably infinite dimension are not all isomorphic. *I. Halperin.*

Sebastião e Silva, J. Le calcul opérationnel au point de vue des distributions. Portugal. Math. **14** (1956), 105–132.

In this paper the author first develops an operational calculus for an operator which is a continuous linear mapping of a sequentially complete locally convex topological linear space E into itself. When E is a certain space of distributions, this operational calculus turns out to be closely related to the use of Laplace transforms on distributions. Some aspects of this situation have already been worked out by L. Schwartz and his associates [Medd. Lunds Univ. Mat. Sem. Tome Supplémentaire (1952), 196–206; Séminaire Schwartz de la Faculté des Sciences de Paris, 1953–54, Exposé 22C; MR **14**, 639; **17**, 764]. The present treatment goes beyond what was done before in the discussion of topological issues, and it uses direct methods, without recourse to dual spaces.

We outline the procedure briefly. For $k=0, 1, \cdots$, $\mathfrak{A}_k$ is the set of functions φ such that $\varphi(z)/zk$ is continuous when $\mathrm{Re}(z)\geqq k$ and holomorphic when $\mathrm{Re}(z)>k$. Each $\mathfrak{A}_k$ is given a norm topology, and $\mathfrak{A}_\omega$ is the inductive limit of the $\mathfrak{A}_k$'s. Let C_k be the line $\mathrm{Re}(z)=k$, directed positively upward. The mapping $\lambda\to(z-\lambda)^{-1}$ of the complex plane into $\mathfrak{A}_\omega$ is denoted by $\mathbf{h}$. If E is a sequentially complete locally convex space, the general form of a continuous linear mapping F of $\mathfrak{A}_\omega$ into E is

$$F(\varphi)=\frac{1}{2\pi i}\int_{C_k} f(\lambda)\varphi(\lambda)\,d\lambda,$$

where $\mathbf{f}$ is called the indicatrix of F and is given by $\mathbf{f}(\lambda)=F[\mathbf{h}(\lambda)]$. Three properties characterizing $\mathbf{f}$ are given. Next, let $\Lambda_C(E)$ be the ring of continuous linear mappings of E into itself, with the topology of simple convergence on E. If $\Theta\in\Lambda_C(E)$ conditions are sought for the existence of a continuous ring homomorphism F of $\mathfrak{A}_\omega$ into $\Lambda_C(E)$ which carries $\varphi(z)=z$ into Θ and $\varphi(z)=k$ into kI. Necessary and sufficient conditions for the existence of such an F are (1) $(\Theta-\lambda)^{-1}$ must be an entire function of λ; (2) $\lambda(\Theta-\lambda)^{-1}$ must be bounded on each half plane $\mathrm{Re}(\lambda)\leqq k$; (3) $\lambda(\lambda-\Theta)^{-1}\to I$ as $\mathrm{Re}(\lambda)\to-\infty$. When these conditions are fulfilled $(\Theta-\lambda)^{-1}$ is the indicatrix of the required F. An operational calculus results from writing $F(\varphi)=\varphi(\Theta)$.

Next the author takes for E the space C_ω^+ of distributions on the real line, with support limited on the left. The topology is obtained by viewing the space as the projective limit of a sequence of spaces, each of which is the inductive limit of a sequence of normed spaces. If D is the differentiation operator, it fulfills the conditions imposed on Θ, so the operational calculus exists for D. If T is a distribution and $\varphi\in\mathfrak{A}_\omega$ it turns out that $\varphi(D)T=\Phi*T$, where the star denotes a product of composition of two distributions. The correspondence between φ and Φ is one-to-one, φ being the Laplace transform of Φ, and being what Schwartz calls a laplacisable distribution.

The paper closes with a brief description of extensions of the correspondence between φ and Φ, in the course of which it appears that objects of a more general sort, called ultra-distributions, are needed. *A. E. Taylor.*

★**Kadison, Richard V. Report on operator algebras.** on order
Report of an international conference on operator theory and group representations, Arden House, Harriman, N. Y., 1955, pp. 4–10. Publ. 387, National Academy of Sciences-National Research Council, Washington, D. C., 1955.
Expository paper.

Ogasawara, Tôzirô. Topologies on rings of operators. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 255–272.

The author discusses the common topologies in rings of operators, his main intention being to give conditions for various ones to be equivalent. He gives new proofs of the properties of the coupling operator defined by Kaplansky [same C. R. **231** (1950), 485–486; MR **12**, 186], Pallu de La Barrière [Bull. Soc. Math. France **82** (1954),

1–52; MR 16, 491], and Griffin [Trans. Amer. Math. Soc. 75 (1953), 471–504; MR 15, 539].

These properties are then used to prove that the equivalence of the ultra strong (weak) and strong (weak) topologies depends on a size restriction on the coupling operator. Finally, he shows that the equivalence of the ultra strong (weak) and ultra weak (strong) topologies is equivalent to the finite dimensionality of the ring.

E. L. Griffin, Jr. (Ann Arbor, Mich.).

See also: Samuel, p. 108; Friedman and Mishoe, p. 129; Maurin, p. 130; Schröder, p. 152; Nadler, p. 181.

Banach Spaces, Banach Algebras

Masani, P. The rational approximation of operator-valued functions. Proc. London Math. Soc. (3) **6** (1956), 43–58.

Let X be a Banach algebra with unit, not assumed commutative. Let f be a function with values in X defined and analytic in an open circular annulus A in the complex plane. Let B, D be closed circular annuli with $B \subset A \subset D$. Theorem: If $(f(z))^{-1}$ exists for all z in A, then for every positive ε there exists a rational function R such that $\|(R(z))^{-1}f(z)-1\|<\varepsilon$ for all z in B, where $(R(z))^{-1}$ exists for all z in D and R has poles only at 0 and ∞. Here "rational function" means a product of polynomials in z with coefficients in X and the reciprocals of such polynomials. The proof is based on the Riemann product integral [cf. F. M. Stewart, Trans. Amer. Math. Soc. **68** (1950), 76–104; MR **11**, 505; and P. Masani, ibid. **61** (1947), 147–192; MR **8**, 321]. The author develops and uses some properties of the product integral taken over contours in the plane. The difficulty in the proof is due, of course, to the non-commutativity of the values of f coupled with the requirement that $(R(z))^{-1}$ exist for all z in D so that it is not sufficient to take for R a suitable partial sum of the Laurent series of f. *J. Wermer.*

Masani, P. The Laurent factorization of operator-valued functions. Proc. London Math. Soc. (3) **6** (1956), 59–69.

Let X be a Banach algebra with unit, not necessarily commutative. Call a function H with values in X "invertible" on a set S if $(H(z))^{-1}$ exists for all z in S. Let A denote the annulus $r<|z|<R$ in the plane and let B be a closed annulus contained in A. Theorem 1: Let F be a function holomorphic and invertible on A with values in X. There exists a neighborhood N of 1 in X (N depending only on A and B) such that if $F(z) \in N$ for all z in B, then we can write

$$F(z)=F_+(z)\cdot F_-(z) \quad (z \in A),$$

where F_+ and F_- are holomorphic and invertible for $|z|<R$ and $|z|>r$ respectively and $F_-(\infty)=1$. The author gives two proofs for this theorem. The first proof reduces the problem to the corresponding factorisation on $|z|=1$ of a function (sufficiently close to 1) with absolutely convergent Fourier series with coefficients in X into factors whose Fourier coefficients vanish for $n<0$ and $n>0$ respectively. The possibility of this factorisation is shown by extending to a general Banach algebra a method used by H. Cartan on $n \times n$ matrix algebras [J. Math. Pures Appl. (9) **19** (1940), 1–26; MR **1**, 312]. The second proof is based on a certain Fredholm integral equation. As a Corollary to Theorem 1, obtained by applying to that theorem the author's result on rational approximation given in the preceding review, the author obtains the following result: if we drop in theorem 1 the assumption that the values of F lie sufficiently near the identity, we get the following decomposition:

$$F(z)=R(z)\cdot F_+(z)\cdot F_-(z) \quad (z \in A),$$

where F_+ and F_- are as in Theorem 1 and R is a rational function. R may be chosen so as to be invertible on a preassigned closed annulus containing A and have poles only at 0 and ∞. *J. Wermer* (Princeton, N.J.).

Yen, Ti. Quotient algebra of a finite AW^*-algebra. Pacific J. Math. **6** (1956), 389–395.

It has been shown by F. B. Wright [Ann. of Math. (2) **60** (1954), 560–570; MR **16**, 375] that if M is a maximal ideal in a finite AW^* algebra with trace, then the ring modulo the ideal is an AW^*-factor. The author observes that in any AW^*-algebra of finite type, the dimension function that always exists yields, for the entire algebra, a function which acts like a trace on every commutative subalgebra. This function can be used to extend Wright's theorem to arbitrary AW^*-finite-algebras. Finally, if all finite AW^*-factors have traces, then all finite AW^* algebras have them. *E. L. Griffin, Jr.*

See also: Fichera, p. 131.

Hilbert Space

★ **Ionescu Tulcea, C. T. Spatii Hilbert [Hilbert Space].** Editura Academiei Republicii Populare Romine, 1956. 283 pp. Lei 10,45.

A systematic text-book (though without formal exercises) with the nine chapter-headings: General properties; examples; continuous operators; Banach algebras and structure theorems; operational calculus; operators; hermitian and self-adjoint operators; operational calculus for normal and self-adjoint operators. Three added notes deal with Banach spaces, the Riesz representation theorem and the lemmas of Zorn and Kneser. There is no index.

Vala, Klaus. Sur la puissance extérieure d'un espace de Hilbert. C. R. Acad. Sci. Paris **242** (1956), 2499–2500.

The author announces results on the exterior power $\Lambda^p H$ of a Hilbert space H. The function D defined on $\Lambda^p H$ by the equation

$$D(x_1 \wedge x_2 \wedge \cdots \wedge x_p,\ y_1 \wedge y_2 \wedge \cdots \wedge y_p)=\det|(x_i,y_j)| \qquad (i,j=1,\cdots,p)$$

can be shown to be a positive definite bilinear form, and under the corresponding metric $\Lambda^p H$ can be completed to a Hilbert space $[\Lambda^p H]$. Let T be a linear transformation of H into itself, and $\Lambda^p T$ its pth exterior power; if T is bounded, $\Lambda^p T$ can be extended to a bounded transformation $[\Lambda^p T]$ of $[\Lambda^p H]$. If T is self-adjoint, self-adjoint and completely continuous, normal, unitary, or a projection, then $[\Lambda^p T]$ has the same property. There is essentially only one metric with these properties; if B is a bilinear function on $[\Lambda^p H]$ such that

$$B(\Lambda^p S z_1, z_2)=B(z_1, \Lambda^p S z_2) \quad (z_1, z_2 \in [\Lambda^p H])$$

for each symmetric transformations S of H, then B is of the form λD. where λ is a complex number.

J. H. Williamson (Belfast).

Citlanadze, È. S. On a conditional extremum of a weakly continuous functional in Hilbert space. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 111–124. (Russian)

Let f and $\phi_1, \cdots, \phi_n$ be functionals on Hilbert space H. The author is concerned with the problem of finding extrema for $f(x)$ under the constraints $\phi_i(x)=a_i$, $i=1, \cdots, n$, $x \in H$. Let $[A]$ be a closed compact homotopy class of sets V on the set N of $x \in H$ satisfying the constraints. $[A]$ by definition must contain along with any A all the sets obtainable from A by deformation on N, and all topological limits of sequences of sets belonging to $[A]$ in the sense of Liusternik and Šnirel'man [Uspehi Mat. Nauk (N.S.) **2** (1947), no. 1(17), 166–217; MR **10**, 624]. Let

$$c=\sup_{A \in [A]} \min_{x \in A} f(x).$$

The author uses his definition of Frechet differential at x [Mat. Sb. N.S. **29(71)** (1951), 3–12; MR **13**, 251] denoted by $L_f x$. He proves under general conditions, including Lipschitz conditions on L_f and $L_\phi x$ on N, weak continuity of $L_f x$, non-vanishing of the Gram determinant $|(L_\phi x, L_\phi x)|$ on N, and non-vanishing of $L_f x$ on N, that there is an x and a set $\lambda_1, \cdots, \lambda_n$ of real numbers such that

$$L_f x=\sum \lambda_i L_\phi x,\ f(x)=c,\ \phi_i(x)=a_i,\ i=1, \cdots, n.$$

J. M. Danskin (Princeton, N.J.).

Fišman, K. M.; and Gel'man, I. V. On a criterion for the finiteness of the defect-index of an Hermitian operator. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 185–187. (Russian)

Let A be a closed Hermitian operator on a Hilbert space H, be defined on the linear manifold $D(A)$, everywhere dense in H, and mapping $D(A)$ onto the linear manifold $R(A)$. Then the following theorem is proved: If $\dim R(A)=n \pmod{D(A)}$ $(n<\infty)$ and the operator A has a real point of regular type α, then the defect-index of A is finite and does not exceed (n, n). *J. F. Heyda.*

Krasnosel'skiĭ, M. A. Stability of the critical values of even functionals on the sphere. Mat. Sb. N.S. **37(79)**, (1955), 301–322. (Russian)

The author presents the proofs for the results announced in Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 957–959 [MR **16**, 490]. *J. M. Danskin.*

Kurepa, Svetozar. Semigroups of unbounded self-adjoint transformations in Hilbert space. Hrvatsko Prirod. Drustvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 233–238. (Serbo-Croatian summary)

For each $t \geqq 0$ let $A(t)$ be a self-adjoint linear transformation on a fixed Hilbert space R, not necessarily separable. Let $D_t \subset R$ be the domain of $A(t)$ and suppose that $\bigcap_t D_t=D$ is dense in R. The $A(t)$ are supposed to form a semi-group in the sense that $A(s)[A(t)f]=A(s+t)f$ if $f \in D$. Let T be a subset of $[0, \infty)$ such that the inner measure of the sum set $T+T=[t; t_1+t_2=t, t_1 \in T, t_2 \in T]$ is positive. Suppose that the supremum of $|(A(t)f, f)|$ on T is finite for each $f \in D$. Then there exists a unique self-adjoint transformation A such that $A(t)=\exp(At)$. The proof is based on the fact that $\log \|A(t)f\|$ is convex and consequently continuous by the properties of T. If in addition $[A(t), -\infty<t<+\infty]$ is a group, then D is contained in the domain of A. In particular, if $A(t)$ is bounded for each t, then A is bounded and vice versa.

E. Hille (New Haven, Conn.).

Feldman, Jacob. Nonseparability of certain finite factors. Proc. Amer. Math. Soc. **7** (1956), 23–26.

Let A be a ring of operators of finite type acting on a Hilbert space. Let Z be the center of A, Γ the space of maximal ideals of Z and f the natural isomorphism from Z onto the space of continuous complex functions on Γ. Let tr denote the center-valued trace on A. For each $\gamma \in \Gamma$ put $(X, Y)_\gamma=f(\text{tr } Y^*X))(\gamma)$. Let M_γ be the two-sided ideal of the elements X of A for which $(X, X)_\gamma=0$ and A the quotient algebra A/M_γ. It has been shown by F. B. Wright [Ann. of Math. (2) **60** (1954), 560–570; MR **16**, 375] that A_γ is a finite AW^* factor with numerical trace given by $f((\text{tr } (A))(\gamma)$. Using this trace to define an inner product on A_γ one completes A_γ and obtains a Hilbert space H_γ. The author studies this Hilbert space and proves that H_γ is nonseparable except when it is trivially separable by giving criteria on γ. From the non-separability of H_γ it follows that A_γ can have no representations by operators on separable Hilbert spaces. *F. I. Mautner.*

See also: Maurin, p. 130; Yen, p. 138; Klee, p. 141.

TOPOLOGY

General Topology

Aleksandrov, A. D. Geometry and topology in the Soviet Union. I. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat. Fiz. (3) **10** (1956), no. 2(17), 5–28. (Romanian)

A translation of the article listed in MR **10**, 261.

Neumer, Walter. Zur Konstruktion von Ordnungszahlen. V. Math. Z. **64** (1956), 435–456.

The constructive algorithm developed in part I [Math. Z. **58** (1953), 391–413; MR **15**, 512] is extended to a larger initial segment of the second number class. The distinguished sequences assigned to the limit numbers of this initial segment are shown to satisfy an auxiliary condition which Bachmann [Comment. Math. Helv. **28** (1954), 9–16; MR **16**, 20] has proved cannot be retained for the entire second number class.

F. Bagemihl (Notre Dame, Ind.).

Cuesta, N. A consequence of the hypothesis $\aleph_1<2^{\aleph_0}$. Rev. Mat. Hisp.-Amer. (4) **16** (1956), 11–14. (Spanish)

If $2^{\aleph_0}>\aleph_1$, then in the class of subsets of the set of natural numbers, partially ordered by set inclusion, there is no maximal chain of power $\mathfrak{n}$ with $\aleph_0<\mathfrak{n}<2^{\aleph_0}$.

F. Bagemihl (Notre Dame, Ind.).

Fraïssé, Roland. Sur quelques classifications des relations, basées sur des isomorphismes restreints. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 15–60 (1956).

L'article est la première partie d'un ouvrage constituant mise au point des considérations antérieures de l'A. et de Possel [cf. MR **15**, 296; **16**, 227, 1006]. L'élément nouveau c'est la notion de m-chaîne de relations. Une m-relation de base E est chaque sous-ensemble de E^m (m cardinal fini). Chaque famille de m-relations R_i, i parcourant un ensemble ordonné I telles que si $i<j$, la relation R_j soit une restriction de R_i ou $R_i=R_j$ s'appelle une m-chaîne. Les considérations relatives aux m-relations (isomorphis-

mes, parentés etc.) se transplantent convenablement aux m-chaînes. L'outil principal dont l'A. se sert c'est celui d'isomorphisme restreint d'une relation vers une autre; R, R' étant m-relations de bases E, E', chaque application biunivoque φ définie sur $F \subseteq E$ et transformant la restriction $R|F$ de R sur F en $R'|\varphi F$ s'appelle un isomorphisme restreint [i.r.] de R vers R'. Comme tel est considéré aussi l'application $\varphi_\emptyset$ de l'ensemble vide $\emptyset$ sur lui-même. Ces isomorphismes sont classifiés par la parenté $\frac{n}{p}$. Soit F le sous-ensemble de E dans lequel l'i.r. de R vers R' est défini; si $p \geqq q = |F|$, on dit que φ est un $\begin{bmatrix} o \\ p \end{bmatrix}$-isomorphisme. Si $p \geqq q$ et si pour chaque $\bar{F}$, $F \subseteq \bar{F} \subseteq E$ ayant $\leqq p$ points, il existe une application biunivoque prolongeant φ à $\bar{F}$ qui est un $\begin{bmatrix} n-1 \\ p \end{bmatrix}$-isomorphisme, et s'il en est ainsi en permutant R, R' et φ, φ^{-1} respectivement, l'A. dit que φ est un $\begin{bmatrix} n \\ p \end{bmatrix}$-isomorphisme. R' est $\begin{bmatrix} n \\ p \end{bmatrix}$-parente de R, symboliquement $R \overset{n}{\underset{p}{\sim}} R'$ si l'application $\varphi_\emptyset$ est un $\begin{bmatrix} n \\ p \end{bmatrix}$-isomorphisme de R vers R'. Si R, R' sont $\begin{bmatrix} n \\ p \end{bmatrix}$-parentes pour chaque entier p, l'A. dit que R, R' sont $\overset{n}{\underset{o}{\sim}}$ parentes: $R \overset{n}{\underset{o}{\sim}} R'$. Si $R \overset{n}{\underset{o}{\sim}} R'$ pour chaque n, l'A. écrit $R \overset{\infty}{\underset{o}{\sim}} R'$. Pour m, n, p entiers donnés, le nombre de $\frac{n}{p}$-classes est fini (Th. I). Si la base E de E est infinie et si $D \subseteq E$ est infinie, il existe un $\bar{D}$, $D \subseteq \bar{D} \subseteq E$ de même puissance que D et tel que $R|D \overset{\infty}{\underset{o}{\sim}} R$ (Th. II). Si la base de R est un cardinal infini π, pour chaque cardinal $\bar{\pi} > \pi$ il existe un prolongement $\bar{R}$ de R tel que $\bar{R} \overset{\infty}{\underset{o}{\sim}} R$ et que la base de $\bar{R}$ soit de puissance $\bar{\pi}$ (Th. V). L'A démontre encore d'autres propositions.

Ses théorèmes sont mis en correspondance avec quelques considérations dans la logique mathématique.

Đ. Kurepa (Zagreb).

Knaster, B. Sur la fixation des décompositions. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 193–196.

According to a result of Kelley [Trans. Amer. Math. Soc. **52** (1942), 22–36; MR **3**, 315] and Kosiński [Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 69–72; MR **16**, 1140], if f is a monotone open mapping of a compact metric space X onto a finite-dimensional space Y, such that the diameters of the sets $f^{-1}(y)$ are bounded away from 0, then given $\varepsilon > 0$ there exists a finite collection $G_1, \cdots, G_k$ of open sets with disjoints closures such that diam $G_i < \varepsilon$ and f maps $\sum G_i$ onto Y. The author gives examples to show that the conclusion can be false for maps which are not open.

E. E. Floyd (Charlottesville, Va.).

Myškis, A. D.; and Vigant, È. I. On a connection of proximity spaces with extensions of topological spaces. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 969–972. (Russian)

A completely regular space R is called an extension of $M \subset R$ if denoting by $R[A]$ the closure of A in R there exists for $A \subset M$, $B \subset M$ a $C < M$ such that $R[A] \cap R[C] = 0$ and $R[B] \cap R[M - C] = 0$; an extension $R \subset M$ is called accessible if, for any $x \in R - M$, there exist $x_n \in M$ with $x_n \to x$. For any proximity space M, a topological space $\tilde{R} = \tilde{R}(M) \supset M$ is defined, points of $\tilde{R} - M$ being "ends" of a special class [cf. Myškis, same Dokl. (N.S.) **84** (1952), 879–882; MR **14**, 1001]. A (rather complicated] necessary and sufficient condition is given under which $\tilde{R}$ is an accessible extension of M. This condition is shown to be equivalent with the following property of uM (the compact extension of the proximity space M): if $A \subset M$, $B \subset M$, $R[A] \cap R[B] \neq 0$, then $R[R'] \cap R[A] \cap R[B] = 0$ where $R' \subset uM$ consists of limits of sequences of points from M; it is proved that, in this case, R' is equivalent to $\tilde{R}(M)$.

M. Katětov (Prague).

Brace, John W. Almost uniform convergence. Portugal. Math. **14** (1956), 99–104.

Let $\{f_\alpha, \alpha \in A\}$ be a net of real or complex valued functions defined on a set S. Sirvint [Studia Math. **11** (1950), 71–94; MR **14**, 183] introduced the notions of almost uniform convergence of such a net. $\{f_\alpha\}$ converges almost uniformly to the function f_0 on a set $T \subset S$ if and only if for every net $\{x_\beta\}$ defined from a directed set B to the set T,

$$\lim_{\alpha} [\liminf_{\beta} |f_\alpha(x_\beta) - f_0(x_\beta)|] = 0.$$

This paper concerns itself with the characterization of weak and weak* convergence on the conjugate X^* of a convex linear topological space X in terms of almost uniform convergence on certain types of subsets of X. The principal theorem states that the weak topology on X^* is equivalent to the topology of almost uniform convergence on bounded subsets of X. This provides a method of characterizing weak convergence on X^* without reference to X^{**}. The paper concludes with several theorems on weak continuity of linear transformations on X. In particular, the above characterization of weak convergence provides a short proof of the weak continuity of the adjoint of a weakly continuous operator.

R. E. Fullerton (College Park, Md.).

Morita, Kiiti. On images of an open interval under closed continuous mappings. Proc. Japan Acad. **32** (1956), 15–19.

The Hahn-Mazurkiewicz characterization of Peano spaces is extended as follows. For a metric space Y to be the image of the real line under a closed continuous mapping, it is necessary and sufficient that Y be connected, locally connected, locally compact (and therefore separable), and have at most 2 end-points in the sense of Freudenthal [Ann. of Math. (2) **43** (1942), 261–279; MR **3**, 315]. More generally, the closed continuous images of the union of a finite number m of open rays with a common origin (but otherwise disjoint) are characterized in the same way but with 2 replaced by m.

A. H. Stone.

Jewett, John. Differentiable approximations to light interior transformations. Duke Math. J. **23** (1956), 111–124.

Le principal résultat de ce mémoire est le suivant: une application intérieure f d'un domaine plan dans R^2 peut être approchée uniformément dans $\bar{D}$ par une application intérieure n fois continûment différentiable et égale à f sur la frontière de D.

Ce résultat est atteint en introduisant une approximation intérieure de f, linéaire dans chaque simplex d'une triangulation de D de maille suffisamment fine. Cette nouvelle application peut à son tour être approchée par une application intérieure une fois différentiable et la répétition du procédé permet d'obtenir une approximation intérieure n fois différentiable.

L. Fourès.

Wada, Junzo. One-to-one continuous mappings on locally compact spaces. Osaka Math. J. **8** (1956), 19–22.

This paper conditions under which a one-to-one

continuous transformation of a locally compact Hausdorff space onto a locally compact Hausdorff space is a homeomorphism. The author proves the following theorem: Let X be a locally compact Hausdorff space which is the union of a countable collection $\{X_i\}$ of bicompact Hausdorff spaces such that for each i, X_i is contained in the interior of X_{i+1}, and $X-X_i$ is connected. Let Y be a locally compact but not bicompact Hausdorff space which is the image of X under a one-to-one continuous mapping f. Let D be the set of all points x of X such that the inverse mapping f^{-1} is not continuous at $f(x)$. Then f is a homeomorphism if there exists a bicompact subset of X which contains the set D. *D. W. Hall.*

Floyd, E. E. Examples of fixed point sets of periodic maps. II. Ann. of Math. (2) **64** (1956), 396–398.

[For part I see Ann. of Math. (2) **55** (1952), 167–171; MR 13, 573.] Let T be a periodic homeomorphism of a finite complex Y. It is known that if the period is a power of a prime p then if Y is homologically trivial over Z_p so is the fixed-point set F; if Y is a homological sphere over Z_p so is F. The author constructs examples showing that these results do not hold when the period is not the power of a prime. In all the examples, the period is 6. In the first example, Y is contractible and F is the disjoint union of an arc and a simple closed curve; in the second, Y is a simply connected 2-sphere over Z and F the disjoint union of two points and a simple closed curve; in the third, Y is a 3-dimensional compact metric absolute retract and F the disjoint union of an arc and an infinite sequence of circles. *P. A. Smith* (New York, N.Y.).

Gottschalk, W. H. Characterizations of almost periodic transformation groups. Proc. Amer. Math. Soc. **7** (1956), 709–712.

Let X be a compact, uniform space and let T be a multiplicative topological group operating as a (right) transformation group on X. The author treats interconnections between almost periodicity, local almost periodicity and equicontinuity of T [for definitions of terms, see Gottschalk and Hedlund, Topological dynamics, Amer. Math. Soc. Colloq. Publ., v. 36, Providence ,R.I., 1955; MR 17, 650]. It is proved that T is almost periodic if and only if T is locally almost periodic and T is distal; the transformation group T is said to be distal on X provided that if $x, y \in X$ with $x \neq y$, then there is an index α of X such that $(xt, yt) \notin \alpha$ for all $t \in T$. It is also proved that if T is locally almost periodic on X, then T is not equicontinuous on X if and only if there exists a pair of distinct points of X which is syndetically proximal under T; here the pair (x, y) is said to be syndetically proximal provided that if α is an index of X, then there exists a syndetic subset A of T such that $(xa, ya) \in \alpha$ for all $a \in A$.

E. E. Floyd (Charlottesville, Va.).

Morita, Kiiti. On closed mappings and dimension. Proc. Japan Acad. **32** (1956), 161–165.

Let f be a closed continuous mapping of a normal space X onto a normal space Y. Two theorems of Hurewicz for separable metric spaces are extended as follows. Theorem 1. If, for each $y \in Y$, $f^{-1}(y)$ has at most $k+1$ points, $\dim Y \leq \operatorname{Ind} X + k$. Some improvements of this result ($\dim Y \leq \dim X + k$, $\operatorname{Ind} Y \leq \operatorname{Ind} X + k$) are obtained in some special cases. Theorem 2. If Y is paracompact T_1 and, for each $y \in Y$, $\dim f^{-1}(y) \leq m$, then $\dim X \leq \operatorname{Ind} Y + m$; again improvements such as $\dim X \leq \operatorname{ind} Y + m$ are possible in special cases. Here $\dim X$ is the covering dimension, $\operatorname{Ind} X$ the dimension defined inductively by separation of closed sets, and $\operatorname{ind} X$ the Menger-Urysohn dimension. *A. H. Stone* (Manchester).

Hofmann, H. Über eine Dimensionstheorie in topologischen Verbänden. Fund. Math. **42** (1955), 289–311.

Dimension theory has been extended to certain topological Boolean algebras by Sikorski [Fund. Math. **38** (1951), 153–166; MR **14**, 70] with a "global" definition of dimension (corresponding to the dimension, for topological spaces, defined inductively in terms of separation of closed sets). Here the definition adopted is a "local" one (corresponding to the Menger-Urysohn dimension). The two are shown to agree for Sikorski's "C-algebras", here renamed "S-Verbände". An S-Verbände is a Boolean σ-algebra with a T_1 regular "classical" topology (in the sense of Nöbeling: Grundlagen der analytischen Topologie) which has a countable open base. For any Boolean algebra $\mathfrak{B}$ with classical topology, the author defines $\dim \mathfrak{B}$ inductively: $\dim \mathfrak{B} = -1$ means $\mathfrak{B}$ consists of 0 only; $\dim \mathfrak{B} \leq n$ means that, given any non-zero $S \in \mathfrak{B}$ and any open U with $U \wedge S \neq 0$ (one may assume $U \geqq S$ here), there exists open $V \leqq U$ such that $V \wedge S \neq 0$ and $\dim(\bar{V} \wedge \bar{c} V) \leq n-1$; finally if $A \in \mathfrak{B}$, $\dim A$ means the dimension of the principal ideal of A in $\mathfrak{B}$. The author shows that many theorems, classical when $\mathfrak{B}$ is the algebra of subsets of a separable metric space (e.g., subspace, sum, decomposition and covering theorems, and Hurewicz's theorems on mappings which raise or lower dimension), and their classical proofs (as in the books of Hurewicz-Wallman and Menger) can be extended to S-Verbände (sometimes more generally) satisfying suitable separation axioms (varying up to complete normality). Imbedding theorems are not considered here (they have been considered by Sikorski, loc. cit.); however, an analogue is found for the "set of points at which the dimension is $\geqq k$", and the expected theorems are shown to hold. *A. H. Stone* (Manchester).

Klee, V. L., Jr. Fixed-point sets of periodic homeomorphisms of Hilbert space. Ann. of Math. (2) **64** (1956), 393–395.

In striking contrast with the known facts about periodic homeomorphisms of finite dimensional spaces, it is shown that in Hilbert space, any compact subset whatever can be the fixed point set of a periodic homeomorphism of pre-assigned period. *P. A. Smith.*

Dal Soglio, Letizia. Grado topologico e teoremi di esistenza di punti uniti per trasformazioni plurivalenti di 3-celle. Rend. Sem. Mat. Univ. Padova **25** (1956), 386–405.

Let T be an upper semicontinuous multivalent mapping $E \to E^3$ where E is a subset of euclidean space E^3. Let E_0 consist sof the points P for which $P \in T(P)$. It is assumed that for each $P \in E - E_0$, $T(P)$ is a finite union of disjoint closed bounded convex sets. There is also an assumption about the components of the sets $T(P)$ which involves an additive group G. {The significance of this assumption is not clear to the reviewer.} When E is a 3-cell there can be defined a degree for T relative to a given 2-cycle. If the degree relative to the boundary 2-cycle of E is different from 0, E_0 is non-empty. The author gives special consideration to the case in which $T(P)$ is finite for each P, E being a 3-cell, and gives conditions for the non-vacuity of E_0. If each $T(P)$ contains at most 2 points, a simple

sharpening of the continuity condition is sufficient.
P. A. Smith (New York, N.Y.).

Scorza Dragoni, Giuseppe. Una dimostrazione dell'ultimo teorema geometrico di Poincaré. Rend. Sem. Mat. Univ. Padova **25** (1956), 1–104.

This paper is devoted to a proof of the following theorem. Let R be a plane annulus bounded by concentric circles α and β. Let T be a homeomorphism of R onto R which admits no fixed points and which moves points on α and β in opposite directions. Then there exists in R a simple closed curve γ which encircles the inner boundary of R and has no point in common with its image $T\gamma$. The proof occupies about 90 pages and contains many references to a paper by the author (not yet published) on fixed-point-free automorphisms of the plane. The paper contains no reference to G. D. Birkhoff who gave a short elegant proof in 1925 [Acta Math. **47** (1926), 297–311]. *P. A. Smith* (New York, N.Y.).

See also: Choquet, p. 120; Bopp, p. 173; Siegel, p. 178.

Algebraic Topology

Berikašvili, N. A. On the generalized duality theorem of Steenrod. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 385–392. (Russian)

Let the polyhedron $|K|$ with triangulation K be an open subspace of the n-sphere S^n, and let $F=S^n\backslash|K|$. The author proves that

$$H^q(K, G)\approx \mathrm{Hom}\{H^p(F, I), G\}+\mathrm{Ext}\{G, H^{p+1}(F, I)\}$$
$$(q>1,\ p+q=n-1),$$
$$H^1(K, G)\approx \mathrm{Hom}\{H^{n-2}(F, I), G\}+\mathrm{Ext}\{G, H^{n-1}(F, I)+I\},$$
$$H^0(K, G)\approx \mathrm{Hom}\{H^{n-1}(F, I)+I, G\},$$

where I is the group of integers, G is a discrete group, $H^q(K, G)$ is the simplicial cohomology group based on infinite cochains (reduced, if $q=0$), and $H^p(F, I)$ is the Čech cohomology group.

To obtain analogous formulas in case $|K|$ is not open in S^n the author considers certain constructions. Let $\{H_\tau, \bar{p}_{\tau\sigma}\}$ be a direct (inverse) system of groups. Then one can define the derived inverse (direct) systems $\{\mathrm{Hom}\{H_\tau, G\}, \bar{p}_{\tau\sigma}\}$ and $\mathrm{Ext}\{G, H_\tau\}, \bar{p}_{\tau\sigma}'\}$ in an obvious way, for any discrete group G. Let M be a topological space and consider the inverse system $\{H_q(n(U), I), \pi_*\}$, $q>0$, where $n(U)$ is the nerve of the covering U of M. Then there are derived direct systems whose limit groups are denoted by $\mathrm{Hom}_q(M, G)$ and $\mathrm{Ext}_q(M, G)$. Certain changes are necessary when $q=0$. Let $\{F_s\}$ be the set of compact subspaces of M. Order the index set $S=\{s\}$ by setting $s<r$ if $F_s\subset F_r$. Then there is an inverse system of groups $\{H^p(F_s, I), l_{sr}\}$ which is defined in an obvious way. The limit groups of the derived direct systems are denoted by $\mathrm{Hom}_{cp}\{M, G\}$ and $\mathrm{Ext}_{cp}\{M, G\}$.

Now let A be an arbitrary subspace of S^n and let $B=S^n\backslash A$. The author proves that

$$\mathrm{Hom}_q\{A, G\}\approx \mathrm{Hom}_{cp}\{B, G\},\ \mathrm{Ext}_q\{A, G\}\approx\mathrm{Ext}_{cp}\{B, G\},$$
$$\bar{H}^q(A, G)\approx\mathrm{Hom}_{cp}\{B, G\}+\mathrm{Ext}_{c(p+1)}\{B, G\},$$

where G is a discrete group, p and q are non-negative integers such that $p+q=n-1$, and $\bar{H}^p(A, G)$ is the Čech cohomology group. *N. Stein* (Ithaca, N.Y.).

Hilton, P. J. Note on the higher Hopf invariants. Proc. Cambridge Philos. Soc. **52** (1956), 750–752.

The higher generalized Hopf homomorphisms

$$\pi_q(S^{n+1})\to\pi_q(S^{kn+1})$$

were introduced by the present author in a previous paper [J. London Math. Soc. **30** (1955), 154–172; MR **16**, 847]. For $k=2$, they reduce to the Hopf homomorphism introduced by G. W. Whitehead [Ann. of Math. (2) **51** (1950), 192–237; MR **12**, 847]. Not much is known about them for the case $k>2$. The reviewer has posed the following problem: Do there exist integers n and q such that these homomorphisms are non-zero for some integer $k>2$ [ibid **62** (1955), 327–359; MR **17**, 653]? In the present paper, the author gives several cases (e.g. $n=1$ and $q=6$, 7, or 8) where the homomorphisms are non-zero for $k=4$. Apparently there are still no examples known where these homomorphisms are non-zero for $k=3$.
W. S. Massey (Providence, R.I.).

Kan, Daniel M. Abstract homotopy. I, II. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 1092–1096; **42** (1956), 255–258.

In these two papers the author starts a study of so called cubical complexes. An abstract cubical complex bears the same relation to the cubical singular complex of a topological space as a semi-simplicial complex does to the total singular complex of a topological space [Eilenberg and Zilber, Ann. of Math. (2) **51** (1950), 499–513; MR **11**, 734]. In particular if $\square_n$ denotes the standard n-cube, then the element of dimension n of a cubical complex K correspond in a natural one-to-one fashion with the cubical maps of $\square_n\to K$.

The main point of the author in these papers is to introduce the study of cubical complexes satisfying the extension condition, in other words cubical complexes K such that if $\square_n'$ is the boundary of $\square_n$ minus some face then any map $f:\square_n'\to K$ can be extended to a map $f:\square_n\to K$. This corresponds to the intuitive geometric idea that if one takes the boundary of a cube and throws away one face, then what is left is a retract of the entire cube. In the context of complexes satisfying the extension condition most of the usual theorems of homotopy are valid. For example, if K is a complex satisfying the extension condition, then homotopy between maps of another complex L into K is an equivalence relation. This allows the definition of homotopy groups of a complex K satisfying the extension condition in the usual fashion as homotopy classes of maps f: $(\square_n, \dot{\square}_n)\to(K, k)$, where $\dot{\square}_n$ is the boundary of $\square_n$, and k is a point, i.e. the subcomplex of K generated by a 0-cube. These groups have the usual properties of the homotopy groups of a topological space, and in fact if one takes the homotopy groups of the cubical singular complex of a topological space, then one obtains exactly the homotopy group of the space.

In the second paper an abstract definition of category with homotopy is given, and several examples are given of such categories including the category of topological spaces, the category of chain complexes, and the category of cubical complex satisfying the extension condition. Aside from this he goes on to extend the definition of homotopy groups from the absolute case to the relative case. This is done in the classical manner. *J. C. Moore.*

Chang, S. C. On Jacobi identity. Acta Math. Sinica **4** (1954), 365–379. (Chinese. English summary)

Let X be an arcwise connected topological space and

α, β, γ be elements of the homotopy groups $\pi_p(X, x_0)$, $\pi_q(X, x_0)$, $\pi_r(X, x_0)$ respectively, where $r \geq q \geq p \geq 2$. By $[\beta, \gamma]$ we mean the Whitehead product and $[\alpha, [\beta, \gamma]]$ the repeated Whitehead product. The author develops the following consequence:

$$(-1)^{r(p+1)}[\alpha, [\beta, \gamma]] + (-1)^{p(q+1)}[\beta, [\gamma, \alpha]] + (-1)^{q(r+1)}[\gamma, [\alpha, \beta]] = 0,$$

namely, the Jacobi identity.

From the author's summary.

Dold, Albrecht. Vollständigkeit der Wuschen Relationen zwischen den Stiefel-Whitneyschen Zahlen differenzierbarer Mannigfaltigkeiten. Math. Z. **65** (1956), 200–206.

Let M^n be an n-dimensional compact differentiable manifold, and let W_i, $i=0, 1, \cdots, n$, denote the Stiefel-Whitney classes (mod 2) of M^n. On the basis of certain results of R. Thom, W. T. Wu has shown that it is possible to determine the W_i from the structure of the cohomology ring mod 2 of M^n and the Steenrod squares in M^n [see Thom's thesis, Ann. Sci. Ecole Norm. Sup. (3) **69** (1952), 109–182, and the references given there; MR **14**, 1004]. Wu has also proved [C. R. Acad. Sci. Paris **230** (1950), 508–511; MR **12**, 42] that the Steenrod squares of the cohomology classes W_i can be expressed as certain polynomials in the W_i.

These two facts when taken together show that certain relations hold among the Stiefel-Whitney classes in any manifold M^n; these relations assert that certain homogeneous polynomials in the W_i's vanish for all manifolds.

R. Thom has raised the question as to whether or not all relations which hold between the Stiefel-Whitney classes for all manifolds M^n are consequences of Wu relations. In the present paper the author gives a partial answer to this question. He proves that if a relation $P(W_1, \cdots, W_n)=0$ holds for all compact, differentiable n-manifolds M^n, where $P(W_1, \cdots, W_n)$ is a polynomial in the W_i's which is homogeneous of weight n, then this relation is a consequence of Wu's relations. The proof uses methods introduced in a paper of R. Thom [Comment Math. Helv. **28** (1954), 17–86; MR **15**, 890]. The case where the polynomial P is homogeneous of weight less than n is apparently still open.

W. S. Massey. (Providence, R.I.).

Libermann, Paulette. Sur les structures presque complexes et autres structures infinitésimales régulières. Bull. Soc. Math. France **83** (1955), 195–224.

This is a general review of the subject of regular infinitesimal structures, in the sense of Ehresmann, i.e., of fiber bundles obtained by restriction of the structure group of the tangent bundle of a differentiable manifold. (Riemannian geometry and Kähler structures are excluded). 1. General definitions, including the notions of connection, (restricted) holonomy group, covariant differentiation, characteristic classes of a bundle, the Lie character of the group of transformations which leave a connection invariant. 2. Almost complex structures, results on spheres leading up to the result of Borel-Serre (only S_2 and S_6 appear), Wu's relations between Stiefel-Whitney-, Chern- and Pontryagin classes, Hirzebruch's results on quaternion projective spaces, etc. 3. Integrability of almost complex structures, methods of de Rham, Ehresmann, Eckmann-Frölicher, notions of pure forms of type (p, q), the C-operator. 4. Almost symplectic (a 2-form Ω of maximum rank) and sympletic ($d\Omega=0$) structures. Operators (on forms) $\tilde{*}$ (Ω-adjoint), L (product with Ω), $\tilde{\Lambda}$; Riemannian metric exchangeable with Ω (almost hermitian structure), $*$ (adjoint), Λ ($\tilde{\Lambda}$ and Λ turn out identical), class of a form, simple form, effective forms, Eckmann-Guggenheimer-Lepage decomposition, harmonic forms, Lee's geometry of Ω, etc. 5. Almost hermitian structure. Lichnerowicz's treatment of pseudo-Kähler spaces (Ω covariant-constant); canonically associated connections, expressions for the Chern classes, role of the Ricci tensor, isotropic and locally homogeneous structures, Chern's locally hermitian and Kähler spaces. 6. Scalar curvature, conformal curvature and torsion. Isotropy implies vanishing of the scalar curvature. Conditions for local conformality with integrable or Kähler structures. 7. Almost hermitian structures of kind k, reduction to the group of transformations leaving $z_1\bar{z}_1+\cdots+z_k\bar{z}_k-z_{k+1}\bar{z}_{k+1}-\cdots-z_n\bar{z}_n$ invariant. Obstruction to finding such a structure subordinated to a given almost complex structure. Associated connections, isotropy, local homogeneity, integrability. 8. Almost quaternion structures (reduction from the (special) unitary group to the symplectic group). The complex projective planes $P_{2n}(C)$ do not admit such structures. Wu's necessary and sufficient condition that a 4-manifold admit such a structure. 9. Other regular infinitesimal structures, corresponding to a number of other more or less standard subgroups of $GL(2n, R)$. Almost paracomplex, corresponding to the group that leaves the involution $x_s \to y_s$, $y_s \to x_s$ ($s=1, \cdots, n$) invariant; almost parahermitian; almost quaternion of second kind. Split manifolds (the group reduced to complex triangular matrices), and their role in algebraic geometry, in particular the flag manifolds. There is a very complete bibliography. *H. Samelson* (Ann Arbor, Mich.).

Ore, Oystein. Studies on directed graphs. II. Ann. of Math. (2) **64** (1956), 142–153.

This is a continuation of a previous paper [Ann. of Math. (2) **63** (1956), 383–406; MR **17**, 1116]. In it the author resumes his exposition of his unified theory of the factors of oriented graphs, and discusses the case of infinite but locally finite graphs. The paper concludes with some special applications, among them a condition for an oriented graph to be decomposable into k subgraphs with the same local degrees. *W. T. Tutte.*

Kudo, Tatsuji; and Araki, Shôrô. On $H^*_*(\Omega^N(S^n); Z_2)$. Proc. Japan Acad. **32** (1956), 333–335.

For any space X, let $\Omega(X)$ denote the space of loops in X, and let $\Omega^N(X)$ be defined inductively by $\Omega^0(X)=X$, $\Omega^{N+1}(X)=\Omega[\Omega^N(X)]$. The main result of this note asserts that the Pontrjagin ring $H_*(\Omega^N(S^n), Z_2)$ with coefficients modulo 2 of the N-times iterated loop space of the n-sphere S^n is a polynomial algebra over the field Z_2 for $0<N<n$; the degrees of the generators are explicitly given.

This theorem is proved by an induction on the integer N. $\Omega^{N+1}(S^n)$ is considered as the fibre of a contractible fibre space over $\Omega^N(S^n)$, and the authors make use of the Serre spectral sequence of this fibration. It is well-known that the spaces $\Omega^N(S^n)$ are H-spaces. The author shows that they have even more structure: $\Omega^N(S^n)$ is an H_{N-1}-space. Roughly speaking, an H_0-space is an ordinary H-space with a 2-sided identity, an H_1-space is an H_0-space in which the multiplication is homotopy commutative,

etc. The space of loops in an H_i-space is an H_{i+1}-space.

The authors show how one can define for any H_N-space X certain homology operations Q_i: $H_q(X, Z_2) \to H_{2q+i}(X, Z_2)$, for $0 \leq i \leq N$. The construction of these operations is dual to the construction given by Steenrod [Ann. of Math. (2) **56** (1952), 47–67; MR **13**, 966] for the Steenrod squares. The inductive proof of the main theorem makes use of these operations in a way analogous to the way in which Serre [Comment. Math. Helv. **27** (1953), 198–232; MR **15**, 643] used the Steenrod squares in the determination of the structure of the cohomology algebras $H^*(\Pi, m; Z_2)$. *W. S. Massey.*

Satake, I. On a generalization of the notion of manifold. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.

A local uniformizing system (l.u.s.) of an open set U in a Hausdorff space consists of an open set $\tilde{U}$ in Euclidean space R^n, a finite group G of linear transformation of R^n (the set of fixed points is assumed of dimension $\leq n-2$) and a homeomorphism of U with the quotient space $\tilde{U}/G$. A C^∞-V-manifold $\mathcal{V}$ of dimension n is a Hausdorff space, with a system of l.u.s.'s which form a base for the topology and satisfy the necessary compatibility conditions. The quotient space of a manifold by a properly discontinuous group is a V-manifold. C^∞-maps and differential forms can be defined. De Rham's theorems are proved by Weil's method, operating with coverings and Čech groups, for paracompact $\mathcal{V}$. Poincaré duality holds for real coefficients (and orientable $\mathcal{V}$); in terms of forms the duality pairing is given by $\int \omega \wedge \eta$ (with suitable definition of integration of an n-form over V). *H. Samelson.*

See also: Rado and Reichelderfer, p. 115.

GEOMETRY

Geometries, Euclidean and other.

Laugwitz, Detlef. Über die Invarianz quadratischer Formen bei linearen Transformationen und das Raumproblem. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1956**, 21–25.

Let G be a group of linear homogeneous transformations which act on a real vector space L of finite dimension. If $\sup |T\xi| < \infty$ for every element $T \in G$ and for every vector ξ such that $|\xi| \leq 1$, then we say that G is bounded. The author first proves the fundamental lemma of this paper. "If the group G is bounded, then there exists a positive definite quadratic form invariant under G". This is a known theorem [W. Maak, Fastperiodische Funktionen, Springer, Berlin; 1950; MR **13**, 29], however, the author's proof is new, simple and geometrical. It depends upon the idea of so-called Loewner's ellipsoid.

In the next place he formulates Helmholtz's problem in the following way: "Suppose that G has the following properties: a) G is bounded. b) For every set of vectors $\xi \in L$ and $\eta \in L$ ($\xi \neq 0$, $\eta \neq 0$), there exists $T \in G$ and a real number $\mu > 0$ such that $T\xi = \mu\eta$ (free mobility of directions). Then, there exists a positive definite quadratic $Q(\xi)$ which is invariant under G and every other invariant is a function of Q. Noteworthy is that the hypotheses of free mobility of elements of higher dimensions are replaced by the boundedness of G. The proof is easily given by virtue of the above lemma.

In the third place he formulates Helmholtz's problem in another way and calls it an isotropy theorem. It is stated as follows: Suppose $F(\xi)$ is a Minkowski metric in L such that $F(\xi) > 0$ for $\xi \neq 0$, $F(a \cdot \xi) = aF(\xi)$ for $a > 0$ and the set of ξ satisfying $F(\xi) \leq 1$ is bounded. Then, in order that F^2 is a quadratic form, it is necessary and sufficient that every invariant of ξ is constant along the indicatrix $F = 1$. The proof is given by virtue of Loewner's ellipsoid again.

These theorems are extended to vector spaces of arbitrary (including infinity) dimension under some modification of the hypothesis.

S. Sasaki (Sendai).

Cuzzer Notari, Vittoria. Sistema fondamentale per la trigonometria e superficie cubica. Period. Mat. (4) **34** (1956), 13–37.

Brisac, Robert. Un exemple de géométrie plane où les points sont en nombre fini, précédé d'une note de René de Possel. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 7–12 (1956).

R. Brisac (1907–1947) est l'auteur d'un „Exposé élémentaire des principes de la géométrie euclidienne" [Gauthier-Villars, Paris, 1955; MR **17**, 774] où il présente un système d'axiomes dans lequel les déplacements jouent un rôle essentiel. L'exemple de géométrie plane ici donné comprend 9 points, 12 droites, 72 glissements et 144 déplacements. Tous les axiomes du système sont satisfaits à l'exception de l'axiome du retournement de l'angle et de l'axiome d'ordre.

M. de Possel qui a dirigé les travaux de Brisac sur la géométrie fait précéder cet exemple des énoncés des axiomes qui interviennent. *M. Decuyper* (Lille).

d'Orgeval, Bernard. Remarque sur la note de M. Brisac. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 13 (1956).

M. d'Orgeval remarque que la géométrie des neuf points envisagée dans l'article précédent est en étroite relation avec celle de la configuration des neuf points d'inflexion d'une cubique plane. Il précise l'interprétation de certains "glissements" de Brisac et signale l'intérêt que présenterait l'étude plus complète de cette analogie.

M. Decuyper (Lille).

Pickert, Günter. Einfacher Beweis eines Satzes von M. Hall über offene Inzidenzstrukturen. Arch. Math. **6** (1955), 417–419.

By an incidence structure the author means what M. Hall calls a partial plane; by an open incidence structure, what Hall calls a finite open configuration. The following theorem is originally due to Hall [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72]: A non-degenerate open incidence structure is free-equivalent, for a uniquely determined integer $n \geq 4$, to an incidence structure consisting of n distinct points and a line through $n-2$ of these. The author gives a proof by induction on the number of elements (points and lines) of the incidence structure. The proof is more elementary than that of Hall (though perhaps no less demanding) in that it reduces to the consideration of eight rather special types of incidence structure. The latter are examined with the help of the following simple lemma, implicitly used by Hall: An

incidence structure which contains the distinct points A_1, A_2, B_1, B_2 and the distinct lines a, b with (for $i=1, 2$) A_i on a but not on b and B_i on b but not on a is free-equivalent to the incidence structure which results on deletion of all points, other than A_1, A_2, which lie only on a and adjunction of a like number of points which lie only on b. *R. H. Bruck* (Madison, Wis.).

Peschl, Ernst. Winkelrelationen am Simplex und die Eulersche Charakteristik. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955,** 319–345 (1956).

Consider a general simplex in n-dimensional spherical space. Let ω_k denote the sum of the angles at its $\binom{n+1}{k+1}$ $(n-k-1)$-cells, each measured as a fraction of the whole angle at the $(n-k-1)$-flat concerned, with the conventions $\omega_{-1}=1$, $\omega_0=\frac{1}{2}(n+1)$, while ω_n is the content of the simplex itself (as a fraction of the whole spherical space). Generalizing a formula of Schläfli [Ges. math. Abh., Bd. 1, Birkhäuser, Basel, 1950, p. 240; MR **11**, 611], the author proves that these sums satisfy the equations

$$\omega_{2l}=\sum_{j=1}^{l+1}\frac{2^{2j}-1}{j}\binom{n-2l+2j-1}{2j-1}B_{2j}\omega_{2l-2j+1}$$

where $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, $\cdots$; $0\leq l\leq\frac{1}{2}n$. He also shows that, with an appropriate reinterpretation of ω_n, the same results hold in hyperbolic n-space.

{The formally simpler relations

$$\sum_{k=r-1}^{n}(-1)^{k+1}\binom{k+1}{r}\omega_k=\sum_{k=n-r}^{n}(-1)^{n-k}\binom{k+1}{n-r+1}\omega_k$$

and

$$\sum_{k=-1}^{p}(-1)^{k+1}\binom{n-k}{n-p}\omega_k=\omega_p$$

were given by D. M. Y. Sommerville [Proc. Royal Soc. London. Ser. A. **115** (1927), 103–119] and Walter Hohn [Winkel und Winkelsumme im n-dimensionalen euklidischen Simplex, Thesis, Zürich, 1953; MR **15**, 55]. The author's relations have the advantage of involving fewer terms.} *H. S. M. Coxeter* (Toronto, Ont.).

Florin, F. Zur Konstruktion der Ellipse mit Hilfe von Krümmungskreisen. Forsch. Gebiete Ingenieurwesens **22** (1956), 134–137.

Given an ellipse E, let r, w denote two of its conjugate semi-diameters, δ- the angle between them, and ϱ the radius of curvature of E at the and of r. The author derives the known formula: $\varrho=w^2/r\sin\delta$ [cf. G. Demartres, Cours de géométrie infinitésimale, Gauthier-Villars, Paris, 1913, p. 20].

If two conjugate diameters of an ellipse are given, both in magnitude and direction, the above formula is used to construct, with ruler and compass, the arcs of the four circles of curvature passing through the ends of the given diameters, and the required ellipse may then be completed using a French curve.

The accuracy of the method may be increased by drawing the parallelogram formed by the tangents to the required curve at the ends of the given diameters. The two pairs of points of the curve lying on the two diagonals of the parallelogram may be determined by a ruler and compass construction, and the two conjugate diameters thus obtained may be treated the same way as the given diameters.

Seven figures illustrate the text. *N. A. Court.*

Stovas, V. M. Geometrical meaning of the first and second eccentricities. Vestnik Leningrad. Univ. **11** (1956), no. 13, 139–141. (Russian)

Let $OE=a$, $OP=b$ be the semi-major and semi-minor axes of an ellipse, and let T, n be the traces on OE, OP of the normal ATn of the curve at a point A having for coordinates $AK=x$, $AL=y$. The author proves the known relation [cf. C. Smith, An elementary on conic sections, by the methods of co-ordinate geometry, new ed., Macmillan, London, 1910, p. 159]:

$$e^2=(a^2-b^2)/a^2=OT/OL.$$

This formula for the eccentricity e of an ellipse is of convenient use in spheriodal geodesy, the ellipse being the meridian of the point A, and $OL=OK=x$ the projection of the radius $\varrho=x$ of the circle of latitude of A upon the equatorial semiaxis OE. From the similar triangles nKA, nOT, ATL we have: $e^2=OT/\varrho=nT/N=On/Kn$, where $N=nA$. The author also defines the "second eccentricity" e_1 as:

$$e_1^2=(a^2-b^2)b^2=e^2/(1-e^2)=OT/TL=nT/AT=On/y.$$

N. A. Court (Norman, Okla.).

Gram, Christian. A remark on the construction of the centre of a circle by means of the ruler. Math. Scand. **4** (1956), 157–160.

Rossier, Paul. Construction au compas, par points, de courbes de genre quelconque. Arch. Sci. Soc. Phys. Hist. Nat. Genève **9** (1956), 102.

Neculce, M. Etude synthétique de la transformation affine. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 179–196. (Romanian. Russian and French summaries)

L'auteur donne construction élémentaire d'une transformation T $(P\to P')$ dans le plan affin où PP' a une direction fixe et il démontre que la transformation affine la plus générale est le produit de T_1 et T_2. *O. Bottema.*

Masotti Biggiogero, Giuseppina. Una osservazione sulla costruzione delle coniche per punti e relative tangenti. Period. Mat. (4) **34** (1956), 8–12.

Muracchini, Luigi. Sulle trasformazioni puntuali di seconda e terza specie fra piani proiettivi. Mem. Accad. Sci. Torino. Cl. Fis. Mat. Nat. (3) **1** (1955), 25–44.

A generic punctual transformation T between two planes determines in a generic point three characteristic directions, namely those directions which maintain the inflectional elements. The transformation T is said to be of second (or third) kind if in a generic (regular) couple of correspondent points two (or all three) characteristic directions concide.

Following Borůvka's methods [Publ. Fac. Sci. Univ. Masaryk **72** (1926); **85** (1927)], in this work the author improves and completes some of the results given by Borůvka, and gives the complete classification of the transformations of second and third kind.

For each type the author determines the fundamental invariants and in several cases gives their geometrical constructions. *C. Longo* (Parma).

Facciotti, Guido. La parabola sulla sfera. Period. Mat. (4) **34** (1956), 38–50.

★ **Lobačevskij, Nikolaj Ivanovič. Nuovi principi della geometria con una teoria completa delle parallele. Saggio introduttivo, traduzione e note di Lucio Lombardo Radice.** Edizioni Scientifiche Einaudi, Torino, 1955. 291 pp.

★ **Kulczycki, Stefan. Geometria nieeuklidesowa. [Non-euclidean geometry.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 189 pp. zł. 10.30.

A semi-popular account, beautifully written and printed, under the three headings: history of geometry; principles of non-euclidean geometry; further development of the theory.

S. H. Gould (Providence, R.I.).

Blanuša, Danilo. Immersion des espaces elliptiques dans des espaces euclidiens à l'aide de coordonnées de Weierstrass. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 181–182. (Serbo-Croatian summary)

The present imbedding of a space El_n into R_N, $N=\frac{1}{2}n(n+3)$ is the same as that of a previous paper [same Glasnik **8** (1953), 3–23, 81–114; MR **14**, 1122; **15**, 467], except for the kind of coordinates used. *A. Nijenhuis.*

Nádeník, Zbyněk. L'élargissement du théorème de Ménélaüs et de Céva sur les figures n-dimensionnelles. Časopis Pěst. Mat. **81** (1956), 1–25. (Czech. Russian and French summaries)

See also: Hussain, p. 105; Rankin, p. 114; Minakshisundaram, p. 136; Ramanathan, p. 136; Katkov, p. 147.

Convex Domains, Integral Geometry

Dinghas, Alexander. Über zwei allgemeine Sätze von Brunn-Minkowski-Lusternikschem Typus. Norske Vid. Selsk. Forh., Trondheim **28** (1955), 182–185 (1956).

If A_1, A_2 are F_σ sets in n-dimensional space let A_0 be the set of points P_0 defined by the vector sums P_1+P_2, $P_1 \in A_1$, $P_2 \in A_2$. f_0, f_1, f_2 are positive-valued Borel measurable functions defined for A_0, A_1, A_2 respectively. r is a fixed positive number. The principal result of the paper is that

$$\Big(\int_{A_0} f_0\,dv_1\cdots dv_n\Big)^{1/(r+n)} \geqq \Big(\int_{A_1} f_1\,dv_1\cdots dv_n\Big)^{1/(r+n)} + \Big(\int_{A_2} f_2\,dv_1\cdots dv_n\Big)^{1/(r+n)},$$

provided that

$$f(P_0)^{1/r} \geqq \sup\{(f_1(P_1))^{1/r}+(f_2(P_2))^{1/r}\},$$
$$P_0=P_1+P_2,\ P_1 \in A_1,\ P_2 \in A_2.$$

For $n=1$ the result has been given by Henstock and Macbeath [Proc. London Math. Soc. (3) **3** (1953), 182–194; MR **15**, 109]. The proof follows by induction. *D. Derry.*

Reads in Stat. Dept 5116157

★ **Jaglom, I. M.; und Boltjanski, W. G. Konvexe Figuren.** Hochschulbücher für Mathematik, Band 24. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. xvi+257 pp. DM 15.00.

A translation by J. Erlebach of the Russian original reviewed in MR **14**, 197.

Ahiezer, N. I. Proof of the multiplier-rule for the isoperimetric problem. Har'kov. Gos. Univ. Uč. Zap. **40**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **23** (1952), 91–93 (1954). (Russian)

Hersch, Joseph. Longueurs extrémales et géométrie globale. Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 401–414.

The author calculates explicitly the extremal lengths of a number of geometrically given curve families and gives some geometrical consequences of their values. {As is the case with several of the author's publications all the basic results of this paper together with most of the derived results were known to Ahlfors and Beurling more than ten years ago. They were presented by Ahlfors in a course at Harvard University in the Spring of 1947. Unfortunately these results were never published in a standard journal although in a set of mimeographed notes [Lectures on conformal mapping, Oklahoma Agric. and Mech. Coll., 1951] Ahlfors gave part of the results. In particular he gave the values of the quantities denoted by the author as $L_{\{e_i\}}$ and $L_{\{\bar{e}_i\}}$ as the author acknowledges in a terminal remark.

The reviewer feels that the author is not justified in applying the result 2·2·3, proved for a single symmetry, simultaneously to an infinity of symmetries in 6 (since the existence of an extremal metric is not a priori assured). It would seem necessary to use some integration process in this connection as was done by Ahlfors and Beurling in calculating the extremal lengths given by the author in this section and by P. M. Pu [Pacific J. Math. **2** (1952) 55–71; MR **14**, 87] in calculating the quantity denoted by the author as $L_{\{n_i\}}$ (although Pu used a different phraseology).

It should be remarked that the author works with a slight variant of the Ahlfors-Beurling definition of extremal length. This appears to provide no essential generalization.

Finally the author fails to make the remark, due to Ahlfors and Beurling, which seems to the reviewer to provide these special extremal length problems with their greatest interest: namely that an extremal metric may exist which is not locally Euclidean.} *J. A. Jenkins.*

Steensholt, Gunnar. Note on an elementary property of triangles. Amer. Math. Monthly **63** (1956), 571–572.

If r_1, r_2, r_3 are the distances of the sides a, b, c of a triangle $(T)=abc$ from a point P inside of (T), and R_1, R_2, R_3 are the distances of P from the respectively opposite vertices of (T), the author obtains, in an elementary way, the relation

$$aR_1+bR_2+cR_3 \geqq 2(ar_1+br_2+cr_3),$$

and points out that this formula proves the Erdös-Mordell formula $R_1+R_2+R_3 \geqq 2(r_1+r_2+r_3)$ for the case when (T) is equilateral $(a=b=c)$. *N. A. Court.*

See also: Birch, p. 114; Rankin, p. 114; Steensholt, p. 146.

Differential Geometry

Berezina, L. Ya. Some bilateral stratified pairs of congruences. Latvijas PSR Zinātņu Akad. Vēstis **1953**, no. 3 (68), 132–134. (Russian. Latvian summary)

The trihedra of reference are selected in such a way

that a set of equations results convenient for certain pairs of congruences. Among them are pairs for which the common perpendicular of corresponding rays passes through the central points of the ray of one of the pairs, pairs for which the bisector of the focal planes of one of the pairs passes through the common perpendicular, pairs for which one is a normal congruence. [Cf. Berezina, Latvijas PSR Zinātņu Akad. Vēstis **1951**, no 8(49), 1317–1325; MR **14**, 1120.] *D. J. Struik* (Cambridge, Mass.).

Cygankov, I. V. Construction of W-congruences with a given focal surface. Molotov. Gos. Univ. Uč. Zap. **8**, no. 1 (1953), 19–23. (Russian)

Given a surface S, to find a W-congruence of which S is a focal surface, if the second focal surface is degenerated into a curve. The solution is obtained with the aid of the theorem that each of the sheets of the focal surface of a W-congruence by infinitesimal bending has its displacement in the direction of the normal to the other sheet. If the second focal surface is a curve (not a straight line), then S is a ruled surface and the problem of finding the corresponding W-congruence depends on a Riccati equation.

D. J. Struik (Cambridge, Mass.).

Urmaev, N. A. Investigations on mathematical cartography. Trudy Central. Nauč.-Issled. Inst. Geodez., Aèros"emki Kartogr. no. **98** (1953), 107 pp. (Russian)

This monograph consists of specialized applications of the differential equations of mathematical cartography together with some associated approximate and numerical methods of integration. The basic equations are developed briefly in a conventional manner. The applications consists principally of projections with equidistant parallels in which cases the specialized forms of the differentional equations are indicated and certain parameters for the projections developed. The second section applies standard methods of integration by use of polynomials and finite differences to the several special cases initially presented. Tables and series with reference to these cases are presented. No bibliography and very limited reference to related literature is given. *N. A. Hall.*

Bilinski, Stanko. Homogene Netze geschlossener orientierbarer Flächen. Bull. Internat. Acad. Yougoslave Sci. Beaux-Arts (N.S.) **6** (1952), 59–75.

A translation of the article reviewed in MR **14**, 1007.

Bakel'man, I. Ya. Estimates of the deformations of regular convex surfaces as dependent on the variation of their inner metric. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 358–361. (Russian)

Given are two regular closed convex surfaces ϕ_0 and ϕ of positive curvature (with or without edge), related by a 1-1 mapping in such a way that at corresponding points the coefficients of the first quadratic forms with their first and second derivatives deviate little from each other. Some estimates are found concerning the deviations from each other of the direction vectors $\bar{r}_0$ and $\bar{r}$ of the surfaces ϕ_0 and ϕ, together with their first and second derivatives, in Cartesian and in spherical coordinates. The problem is essentially one in non-linear differential equations of the elliptic type; some results of J. Schauder [Math. Ann. **106** (1932), 661–721; Math. Z. **38** (1934), 257–282] are utilized.

D. J. Struik (Cambridge, Mass.).

Cypkin, M. E. Differential geometry of the line complex. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 89–107. (Russian)

The lines of E_3 are mapped on the points of a quadric Q_4 in a projective P_5 in accordance with the method developed by A. P. Norden and the author [Dokl. Akad. Nauk SSSR (N.S.) **86** (1952), 23–26; MR **14**, 408]. A line complex $x=x(a_1, a_2, a_3)$, x Plücker coordinates, determines the "coordinate "screws $x_i=dx/da^i$, the tensor $\gamma_{ij}=\gamma_{ji}=x_ix_j$, where $\gamma_{ij}da^ida^j=2dldq$, the tensor $\mathring{\gamma}_{ij}=\mathring{\gamma}_{ji}=x_i\bar{x}_j$, and the tensor $\pi_{ij}=\pi_{ji}=n\partial_jx_i$, where the screw n is defined as a "normal" by $nx=n\bar{x}=nx_i=0$, $n^2=\varepsilon=\pm1$. The tensor γ_{ij} is called the first metrical tensor of the complex, the tensor π_{ij} its second metrical tensor, the tensor $\mathring{\gamma}_{ij}$ the tensor of its spherical representation, since $\mathring{\gamma}_{ij}da^ida^j=dxd\bar{x}=dq^2$. When the ∂_jx_i are decomposed with respect to the basic screws x, $\bar{x}$, x_i, n:

$$\partial_jx_i=\Gamma_{ij}{}^kx_k+\varepsilon\pi_{ij}n-\mathring{\gamma}_{ij}x-\gamma_{ij}\bar{x},$$

then a covariant differentation can be defined by

$$\nabla_jx_i=\varepsilon\pi_{ij}n-\mathring{\gamma}_{ij}n-\gamma_{ij}\bar{x}=\partial_jx_i-\Gamma_{ij}{}^kx_k,$$

for which $\nabla_k\gamma_{ij}=0$. Thus a differential geometry with corresponding curvature tensor can be built up, in which $\Delta v=\partial v^ix_i$; δ, the covariant differential with respect to γ_{ij}, is an interior differential of the screw v. With this apparatus an analogy with ordinary differential geometry is established, in which for the ruled surfaces $x=x(s)$ of the complex the Frenet formulas hold:

$$\frac{\Delta t_0}{ds}=\varepsilon_0\alpha t_1,\ \frac{\Delta t_1}{ds}=\varepsilon_1(-\alpha t_0+\beta t_2),$$

$$\frac{\Delta t_2}{ds}=-\varepsilon_2\beta t_1,\ t_0=\frac{dx}{ds},$$

where s is a parameter so selected that $t_0t_0=1$. There are also sections on invariant lines, on "line-like" and "plane-like" surfaces ($\alpha=0$, resp. $\beta=0$), on singular surfaces ($\bar{a}dx=\bar{a}d^2x=\bar{a}n=0$, a being a tangent screw), on "curvature surfaces" and "asymptotic surfaces", the latter defined by the equation $\pi_{ij}v^iv^j=0$. *D. J. Struik.*

Hua, Loo-Keng. On non-continuable domain with constant curvature. Acta Math. Sinica **4** (1954), 317–322. (Chinese. English summary)

The following theorem is proved: Every bounded non-continuable domain with constant curvature can be mapped pseudoconformally into a unit sphere.

Author's summary.

Katkov, G. F. The affine gradient of a scalar field in an equiaffine space. Trudy Voronež. Gos. Univ. Fiz.-Mat. Sb. **33** (1954), 53–55. (Russian)

This investigation is carried out in the vector-doublet notation of Ya. S. Dubnov [Trudy Sem. Vektor. Tenzor. Analizu **8** (1950), 106–127; MR **13**, 776]. A surface is determined both by its direction vector $\bar{r}=\bar{r}(u^i)$, $i=1, 2$, and its tangent doublet $\bar{\bar{r}}=\bar{r}(u^i)$, systems of reference are determined by $\bar{r}_\alpha\bar{r}^\beta=\delta_\alpha{}^\beta$ (vector system), $\bar{r}_\alpha\bar{\bar{r}}^\beta=\delta_\alpha{}^\beta$ (doublet system). If on a surface we pass through a point M a curve l not on a level surface $\varphi(u^1, u^2, u^3)=C$ of a potential φ, then the tangent vector $d\bar{r}/d\varphi$ and the doublet $\bar{\bar{g}}=\bar{\bar{r}}^\beta\varphi_\alpha$ are related by $\bar{\bar{g}}d\bar{r}/d\varphi=1$. Since in Euclidean space $\bar{r}^\alpha\varphi_\alpha$ is a gradient, in equiaffine space $\bar{\bar{g}}$ is called an affine gradient. Then $\bar{r}_\alpha\bar{\bar{g}}=\varphi_\alpha$, a condition necessary and sufficient for a potential. *D. J. Struik* (Cambridge, Mass.).

Blank, Ya. P. Conical nets. Har'kov. Gos. Univ. Uč. Zap. **40**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **23** (1952), 113–141 (1954). (Russian)

A conjugate system on a surface is a conical net, if it is formed by contact curves of circumscribed cones. Surfaces with such nets were first studied by K. M. Peterson [Mat. Sb. **1** (1866), 391–438] their equations can be written $\varrho x_i=u_i(u)+v_i(v)$, $i=1, 2, 3, 4$, the parameter lines forming the conical net and the vertices of the cones lying on the curves $\varrho x_i=du_i/du$, $\varrho x_i=dv_i/dv$. Later work in this field was done by the author [Soobšč. Har'kov. Mat. Obšč. (4) **11** (1935), 55–68] and by B. Gambier [Ann. Sci. Ecole Norm. Sup. (3) **55** (1938), 83–118; J. Math. Pures Appl. (9) **19** (1940), 63–82; MR **1**, 269]. The author now returns to this subject and studies surfaces with a continuum of conical nets, dealing successively with the case of ∞^4, ∞^3 and ∞^1 such nets. One of the results is that the surfaces with ∞^2 nets are ruled surfaces which, but for real or imaginary collineations, belong to one of the types:

$$(1)\ z=\left(\frac{y}{x}\right)^n, \qquad (2)\ z=\tan^{-1}\frac{y}{x},$$

$$(3)\ z=\frac{y}{x}+x^2, \qquad (4)\ z=\frac{y}{x}+\ln x.$$

D. J. Struik (Cambridge, Mass.).

Vedernikov, V. I.; and Tihonov, V. A. A metrical characterisation of the fundamental forms and quantities of the conformal theory of surfaces. Trudy Voronež. Gos. Univ. Fiz.-Mat. Sb. **33** (1954), 43–52. (Russian)

A surface $x=x(u^1, u^2)$ in Möbius space M_3, x in pentaspherical coordinates, is normalized according to A. P. Norden [cf. Izv. Akad. Nauk SSSR. Ser. Mat. **14** (1950), 105–122; MR **12**, 54]. At every point there exists an orthogonal "normalizing circle", intersection of the two spheres $y_i=\partial_i x-l_i x$, where l_i is the "normalisator". An arbitrary tangent sphere ξ, together with the y_i and the points x and X in which ξ intersects the normalizing circle, form a system of reference, for which by further specialization the equations hold:

$$\partial_i x=y_i+l_i x,\ \nabla_j y_i=l_j y_i+p_{ji}x-g_{ij}x+b_{ij}X,\ \partial_i\xi=-b_{ik}g^{kl}y_l,$$
$$\partial_i X=-p_{ik}g^{kl}y_l-l_i x.$$

A metric is defined by $g_{ij}du^i du^j=0$. There is another system of reference, formed by ω, $\partial_i\omega$, x, X, where ω is a central sphere and x and X are the focal points of the spheres ω, $xX=1$. For this system $\partial_i x=-g^{rs}b_{ri}\omega_s+l_i x$, $\partial_i X=-g^{rs}a_{ri}\omega_s-l_i X$, $\nabla_j\omega_i=a_{ij}x+b_{ij}X-g_{ij}\omega$. With these systems enters a third based on a normalization in which all normalizing circles pass through a fixed point X_0 of M_3, by which an affine E_3 is induced. Here $y_i=\partial_i x$, $\nabla_{ij}x=-\gamma_{ij}X_0+\pi_{ij}\xi$, $\partial_i\xi=-\pi_{ik}\gamma^{kl}\partial_l x$, ξ is now a tangent plane in the E_3, γ_{ij} the metrical tensor, and π_{ij} the tensor of the asymptotic net. Such nets are studied, as well as "minimal surfaces" ($g^{ij}a_{ij}=0$), isothermic surfaces ($\epsilon^{ij}l_{ij}=0$), and Dupin cyclids. *D. J. Struik* (Cambridge, Mass.).

Vedernikov, V. I. Conformal applicability of surfaces with preservation of conjugate geometries. Trudy Voronež. Gos. Univ. Fiz.-Mat. Sb. **33** (1954), 37–42. (Russian)

Such applicability of two surfaces S and $\bar{S}$ exists if each can be normalized in a way [cf. V. I. Vedernikov, Dokl. Akad. Nauk SSR (N.S.) **73** (1950), 437–440; MR **12**, 442] that the corresponding pairs of intrinsic geometries coincide with each other. Necessary and sufficient condition is that a 1-1 correspondence can be set up such that it is a conformal one and that the curvature nets on S and $\bar{S}$ are turned with respect to each other at an angle independent of the particular point. To each surface there exists a family of surfaces depending on four arbitrary functions of one argument which are conformally applicable in the sense indicated. *D. J. Struik.*

See also: Blanuša, p. 146; Sieker, p. 161; Scheffers, p. 179.

Riemannian Geometry, Connections

Green, L. W. Geodesic instability. Proc. Amer. Math. Soc. **7** (1956), 438–448.

The author generalizes to arbitrary dimensions a result he obtained previously for surfaces [Trans. Amer. Math. Soc. **76** (1954), 529–546; MR **16**, 70] dealing with the effect of a geometric assumption, namely the nonexistence of conjugate points, on the geodesic flow and divergence of geodesic rays from a point on a Riemannian manifold M. The principal theorem states that if the sectional curvatures of M are uniformly bounded away from negative infinity and if no geodesic of M contains two mutually conjugate points, then the geodesic rays from each point are uniformly divergent, i.e. as we move out along a geodesic ray our distance from any other geodesic ray with the same initial point must become infinite, in fact uniformly so. This theorem is proved with the help of a detailed investigation of properties of solutions of the Jacobi equation of geodesic variation, a use of special coordinates, and some interesting inequalities which lead to a contradiction if the theorem is assumed false. As easy consequences the author derives the following theorems. (1) If R is compact, has an abelian fundamental group, and has a manifold M with the properties above as universal covering space, then at any point P of R the unit vectors which are tangent to periodic vectors are dense in the space of unit tangent vectors at P. (2) Let M be the interior of the unit ball U_n in euclidean n-space and let $ds^2=4f^2(x)dx_i dx_i/((1-x_i x_i)^2$ be a metric on U_n invariant under a Fuchsian group G of the first kind (i.e. ceases to be properly discontinuous at every boundary point of U_n) which also leaves the hyperbolic metric on U_n ($f(x)\equiv 1$) invariant. Suppose further that the sectional curvatures of this metric are $\geqq -A^2$ and that no geodesic in U_n/G has mutually conjugate points. Then the geodesic flow in U_n/G is topologically transitive. This improves a result of Salenius [Ann. Acad. Sci. Fenn. Ser. A. I. no. **47** (1948); MR **10**, 147] generalizing results of Morse and Hedlund [Trans. Amer. Math. Soc. **51** (1942), 362–386; MR **3**, 309]. *W. M. Boothby* (Evanston, Ill.).

Clark, R. S. On conformal equivalence of Riemannian manifolds which admit an exterior form. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 198–203.

If a Riemannian n-manifold carries an exterior p-form f, which in terms of its coefficients $f_{i_1\cdots i_p}$ in a coordinate neighborhood satisfies $f_i{}^{k_1\cdots k_{p-1}}f_{jk_1\cdots k_{p-1}}=g_{ij}$, g_{ij} being the metric tensor, then it is called a $V(n, p)$. If the form is respectively closed, co-closed, harmonic it is called an $S(n, p)$, $S^*(n, p)$, $H(n, p)$. If the covariant derivative of the corresponding skew-symmetric tensor, i.e. $f_{i_1\cdots i_p,}$ is zero,

it is a $K(n, p)$ and if $R_j^i{}_{kl}=f_j^{ih_1\cdots h_p}S_{klh_1\cdots h_p}$ it is a $G(n, p)$. Examples of some of these with $p=2$ occur in the case of almost complex, symplectic and Kähler structures; a Lie group is a $G(n, 3)$. After making these definitions the author defines certain conformal invariants of a $V(n, p)$ in terms of which he gives necessary and sufficient conditions that a $V(n, p)$ be conformally an $S(n, p)$, $S^*(n, p)$, etc. Most of these are valid not only locally but globally also if the fundamental group of the manifold contains no element of infinite period. *W. M. Boothby.*

Berger, Marcel. Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. Bull. Soc. Math. France **83** (1955), 279–330.

The author gives detailed proofs of results announced in C. R. Acad. Sci. Paris **237** (1953), 472–473, 1306–1308; **238** (1954), 985–986 [MR **15**, 61, 468, 647]. The most striking consequence of his classification is that the holonomy group of an irreducible, non-symmetric Riemannian manifold must act transitively on the unit sphere in the tangent space, posing as problem the proof of this directly, geometrically.

In addition, a list is given of the real irreducible linear Lie algebras that might occur as the Lie algebra of the homogeneous holonomy group of an irreducible affine connection without torsion, however with some details in the non-(Riemannian or indefinite) metric case left out. This paper determines only the holonomy groups for the non-(Cartan) symmetric connections, with some indications as to how the method applies to the symmetric case, which can however be handled more efficiently by means of the theory of homogeneous spaces, an approach the author has carried our elsewhere [ibid. **240** (1955), 2370–2372; **241** (1955), 1696–1698; MR **17**, 644].

The main tool in the non-symmetric case is the study of the curvature tensor and its non-zero covariant derivative. Since by a suitable choice of notation the matrices determined by their first two indices must belong to the Lie algebra of the holonomy group, the Bianchi identities and E. Cartan's theory of the structure of irreducible real linear Lie algebras serve to eliminate all but a short list of possibilities. Apparently it is not known if these possibilities can actually occur as holonomy groups of torsion-free connections. Proofs are given of some applications announced in the notes. For example, using the above main results, the author determines the non-restricted homogeneous holonomy groups of irreducible non-symmetric Riemannian spaces and proves the following theorem: If there exists a differential form invariant by parallel transport on a Riemannian manifold, the metric must be symmetric, reducible or pseudo-Kählerian (Kählerian, if the metric is real analytic). *R. Herman.*

Singh, Kamala Devi. Subspaces of a space with torsion. Tensor (N.S.) **6** (1956), 6–14.

The author discusses what is claimed to be new operations of tensor derivation but such operations have in fact been used since 1925 [e.g. A. Einstein, S.-B. Preuss. Akad. Wiss. **1925**, 414–419; J. M. Thomas, Proc. Nat. Acad. Sci. U.S.A. **12** (1926), 187–191].

A set of curvature tensors is obtained, one corresponding to each definition of tensor derivative; the Mainardi-Codazzi relations are obtained for a particular tensor derivation leading to well-known results to be found, for example, in Eisenhart's "Non-Riemannian geometry" [Amer. Math. Soc. Colloq. Publ., v. 8, New York, 1927]. *T. J. Willmore* (Liverpool).

See also: Dold, p. 143; Satake, p. 144; Remmert, p. 149.

Complex Manifolds

Remmert, Reinhold. Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes. C. R. Acad. Sci. Paris **243** (1956), 118–121.

Zu Anfang der Note werden Einbettungsprobleme für komplexe Räume behandelt und folgende Sätze angekündigt: 1) Jeder n-dimensionale holomorph-separable Raum ist eineindeutig holomorph in den komplexen Zahlenraum C^{2n+1} abbildbar. 2) Ist X ein holomorph-vollständiger komplexer Raum, so gibt es eine lokal-irreduzible analytische Menge V in einem C^N (N hinreichend gross), die zu X analytisch äquivalent ist. Insbesondere ist der Einheitskreis der komplexen z-Ebene im C^3 ohne Singularitäten einbettbar.

Der Abschnitt 3 handelt von Reduktionsproblemen für komplexe Räume. Ein Paar (X^*, τ) heisst der Kern eines komplexen Raumes X, wenn folgendes gilt: 1) X^* ist ein holomorph-separabler Raum, τ: $X\to X^*$ ist eine holomorphe eigentliche Abbildung von X auf X^*. 2) τ induziert in natürlicher Weise einen Isomorphismus des Ringes der in X^* holomorphen Funktion auf den Ring der in X holomorphen Funktionen. Der Verf. gibt an: Jede holomorph-konvexe komplexe Mannigfaltigkeit X besitzt einen Kern (X^*, τ). X^* ist ein holomorph-vollständiger komplexer Raum der Dimension s, wenn auf X genau s analytisch unabhängige Funktionen existieren. Analoges gilt für n-dimensionale holomorph-konvexe Räume, auf denen mindestens $n-1$ unabhängige holomorphe Funktionen vorhanden sind. Als Folgerungen werden u.a. erwähnt: a) Jede (zusammenhängende) holomorph-konvexe komplexe Mannigfaltigkeit besitzt eine abzählbare Topologie. b) Der Approximationssatz von Weil-Oka gilt für alle holomorph-konvexen komplexen Mannigfaltigkeiten. *H. Grauert* (Münster).

Heinz, Erhard. Ein elementarer Beweis des Satzes von Radó-Behnke-Stein-Cartan über analytische Funktionen. Math. Ann. **131** (1956), 258–259.

Ein neuer Beweis (mittels des Poissonschen Integrals) des folgenden Satzes: Es sei $\mathfrak{G}$ eine komplexe Mannigfaltigkeit, $f(z_1, \cdots, z_n)$ eine auf $\mathfrak{G}$ komplexwertige stetige Funktion, welche für alle $z \in \mathfrak{G}$, wo $f(z_1, \cdots, z_n)\neq 0$ ausfällt, holomorph ist. Dann ist $f(z_1, \cdots, z_n)$ in ganz $\mathfrak{G}$ holomorph. *H. Behnke* (Münster).

See also: Berger, p. 149.

Algebraic Geometry

Wilson, E. M. Some formulae for multisecants. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 113–139 (1956). (Turkish summary)

In this paper the author's aim is to generalize well-known formulae by giving and applying a general method for finding formulae which express, in terms of standard characters, the numbers of multisecant linear spaces of algebraic manifolds in higher spaces, whenever these numbers are finite. The correctness of the formulae depends upon the validity of certain assumptions, and therefore no great degree of rigor can be obtained. The author shows their correctness in a large number of cases,

giving examples in which the formulae can be applied. But he does not justify those cases in which his formulae are not valid, cases in which there exist abnormalities sufficient to make the relevant formula inapplicable, abnormalities very often subtly concealed (and for which the author admits the necessity of further investigation). In other words the inapplicability of the formula depends generally as in all researches of this kind, upon particular characters of the manifolds investigated. The author expresses the hope that, in some sense, the formulae, given in this paper, may be true in general, but the "experimental" justification given by him does not permit to conclude this to be absolutely so. — The method and the examples given should be of some interest, and can be useful for further investigations.

M. Piazzolla Beloch (Ferrara).

Godeaux, Lucien. Surfaces dont le système canonique contient quatre composantes fixes. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 764–770.

Scafati, Maria. Sulla classificazione delle superficie ellittiche con un fascio ellittico di curve di genere quattro. Rend. Circ. Mat. Palermo (2) **4** (1955), 367–394 (1956).

L'A. in due precedenti lavori [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16** (1954), 721–724; Rend. Mat. e Appl. (5) **14** (1955), 289–337; MR **16**, 741; **17**, 192] aveva iniziato la classificazione delle superficie ellittiche con un fescio ellittico $\{C\}$ di curve di genere quattro, esaurendo la questione nel caso in cui le curve C sono iperellittiche. Nella presente Memoria l'A. ottiene altresì la classificazione delle superficie ellittiche con un fascio ellittico $\{C\}$ di curve di genere quattro non iperellittiche. Lo scopo è raggiunto tramite le determinazione dei gruppi di trasformazioni birazionali, ciclici o abeliani a base due, appartementi a curve non iperellittiche di genere quattro.

F. Gherardelli (Firenze).

d'Orgeval, B. A propos des surfaces elliptiques avec un faisceau de courbes de genre 4. Publ. Sci. Univ. Alger. Sér. A. 2 (1955), 5 (1956).

„Il m'est un agréable devoir de reconnaître l'antériorité des travaux de Mlle. Scafati." [Quotation from the article]. Editor's note: cf. B. d'Orgeval, Publ Sci. Univ. Alger. Sér A, **1** (1954), 313–336 (1955) [MR **17**, 664] with M. Scafati, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 721–724 (1954) [MR **16**, 741] and Rend. Mat. e Appl. (5) **14**, 289–337 (1955) [MR **17**, 192].

Scott, D. B. On base points of polar curves. Ann. Mat. Pura Appl. (4) **41** (1956), 73–75.

If (K) is the net of polar curves of a plane curve C, then any proper base point of (K) necessarily lies on C; but an improper (infinitely near) base point of (K), as is here pointed out, need not lie on C. Thus, for example, if C is the curve $y(y^2-x^3)+x^5=0$, then C contains the third neighbour of the origin on the branch $y=x^{3/2}$; but the general polar curve of C contains the third neighbour of the origin on the branch $y=3^{-1/2}x^{3/2}$, a point which does not lie on C. More generally, if (C) is a linear system complete with respect to its base, then the system of polar curves of all curves of (C) may have improper base points which are not base points of (C). The same situation is envisaged in relation to an ideal P of curves with assigned multiplicities at a certain set of points and the polar ideal P' generated by all the polar curves of curves of P.

J. G. Semple (London).

Matsusaka, Teruhisa. Polarized varieties, the fields of moduli and generalized Kummer varieties of Abelian varieties. Proc. Japan Acad. **32** (1956), 367–372.

Soit V une variété algébrique complète sans singularités en codimension 1. Une polarisation sur V est un ensemble $\mathfrak{X}$ de diviseurs positifs sur V tel que: a) si $X \in \mathfrak{X}$ alors X' est un diviseur positif pour lequel il existe des entiers m, m' tels que $mX \equiv m'X'$ (équivalente algébrique), si et seulement si $X' \in \mathfrak{X}$; b) $\mathfrak{X}$ contient des diviseurs linéairement effectifs (c'est à dire tels que le plongement projectif de V défini par le système linéaire complet de ce diviseur soit birégulier). Une variété polarisée est une variété V munie d'une polarisation; une variété projective est „naturellement polarisée" si elle est munie de la polarisation $\mathfrak{X}$ contenant ses sections hyperplanes. Soit $\bar{V}$ une variété polarisée par $\mathfrak{X}$; pour $X \in \mathfrak{X}$ et linéairement effectif, soit f_X le plongement projectif de V défini par le système linéaire complet de X; on note $P(V, \mathfrak{X})$ l'ensemble des variétés $f_X(V)$ où V parcourt l'ensemble des diviseurs X de $\mathfrak{X}$ tels que tout $X' \equiv X$ soit linéairement effectif. Si F est une sous famille algébrique de $\mathfrak{X}$, stable par équivalence algébrique, dont tous les systèmes linéaires complets ont même dimension, et contenant des diviseurs X tels que tout $X' \equiv X$ soit linéairement effectif, on note $S(V, F)$ l'ensemble des variétés $f_X(V)$ où X parcourt F. Si V est une variété abélienne, toutes les variétés de $S(V, F)$ sont projectivement équivalentes; on en déduit que le groupe des automorphismes d'une variété abélienne polarisée est finie. Si la caractéristique est 0, ou si V est une courbe, il existe une famille F telle que toutes les familles $S(V, F')$ aient des corps de définitions contenant le corps de définition K de $S(V, F)$; le corps K est alors contenu dans tous les corps de définitions des variétés $W \in P(V, \mathfrak{X})$; on dit que K est le corps des modules de la variété polarisée V. Étant donnée une variété complète absolument normale U et un groupe fini G d'automorphismes birégulier de U, l'auteur construit une variété quotient V de U par G; quand U est une variété abélienne polarisée et G son groupe d'automorphismes, la variété quotient V s'appelle une variété de Kummer généralisée de U. Si la caractéristique est nulle ou si V est une courbe, et si d'autre part le groupe G de tous les automorphismes birégulier de V est fini, une variété quotient W de V par G est définie sur le corps des modules K de V; si $U \in P(V, \mathfrak{X})$ est définie sur un corps K' contenant K, il existe une application canonique définie sur K' de U sur W; ces applications g_U sont compatibles avec les isomorphismes entre les différents éléments U de $P(V, \mathfrak{X})$.

P. Samuel.

Barker, C. C. H. Contact along a curve on an algebraic fourfold. J. London Math. Soc. **31** (1956), 259–268.

This paper deals with residual intersections of primals and surfaces, on an algebraic irreducible and non-singular fourfold V_4, touching along a non-singular curve C and having on C as their only singularities a certain number of nodes. It continues a previous investigation of the author on the same subject [same J. **30** (1955), 343–350; MR **17**, 86], and makes a convenient use of a dilatation of V_4 with base C [cf. B. Segre, Ann. Mat. Pura Appl. (4) **33** (1952), 5–48; MR **14**, 683]; also the special case when V_4 is a linear space is considered.

For instance, one of the results obtained here expresses a canonical set ξ of the curve residual intersection of three primals A, B, D of V_4, touching mutually along C, by means of the equivalence

$$\xi \equiv (ABC \cdot A+B+C+X)-10(C \cdot A+B+D) -16(CX)+12\lambda-6\mu,$$

where X denotes a canonical primal of V_4, λ is a canonical set of C, and

$$\mu\equiv(CA)-\sigma\equiv(CB)-\tau\equiv(CD)-\nu,$$

σ, τ, ν being the sets of the nodes possessed by A, B, D respectively on C. *B. Segre* (Roma).

See also: Satake, p. 144.

NUMERICAL ANALYSIS

Numerical Methods

Luke, Yudell L. Remarks on the τ-method for the solution of linear differential equations with rational coefficients. J. Soc. Indust. Appl. Math. **3** (1955), 179–191.

Lanczos has shown that Chebyshev polynomials can be used effectively to economize Taylor series solutions of linear differential equations with rational coefficients. This note presents an expression which facilitates evaluation of the coefficients in the economized series expansion. The author employs Lanczos τ-method and discusses both homogeneous and non-homogeneous equations for the cases where the formal power series solution has coefficients satisfying either two-term or three-term recurrence relations. *W. E. Milne* (Corvallis, Ore.).

Geissler, D. Entwicklungen von Coulomb-Wellenfunktionen für hohe Energien. Z. Naturf. **11a** (1956), 598–604.

L'équation différentielle

$$\frac{d^2y}{d\varrho^2}+\Big[1-\frac{2\eta}{\varrho}-\frac{L(L+1)}{\varrho^2}\Big]y=0$$

admet deux solutions linéairement indépendantes F_L et G_L telles que

$$F_L(\varrho,\eta)=0 \text{ si } \varrho=0,$$

$$\lim_{\varrho\to\infty} F_L(\varrho,\eta)\to\sin\Theta_L,\ \lim_{\varrho\to\infty} G_L(\varrho,\eta)\to\cos\Theta_L,$$

où

$$\Theta_L=\varrho-\eta\log 2\varrho-L\frac{\pi}{2}+\arg\Gamma(i\eta+L+1).$$

A côté des tables de valeurs numériques publiées par Stegun et Abramowitz [Phys. Rev. (2) **98** (1955), 1851–1852; MR **16**, 1155] pour les fonctions F_L, G_L, $dF_L/d\varrho$, $dG_L/d\varrho$, l'auteur donne ici des développements, en puissances de η, de F_L (valables pour les petites valeurs de ϱ), de F_0 et G_0 (valables pour les grandes valeurs de ϱ). Il y parvient en remarquant que d'une part, F_L est exprimable sous la forme

$$C_L\varrho^{L+1}e^{i\rho}F(L+1+i\eta,\ 2L+2,\ -2i\varrho)$$

avec

$$C_L=\frac{2^L}{(2L+1)!}[L^2+\eta^2]^{\frac12}\times[(L-1)^2+\eta^2]^{\frac12}\times\cdots\times[1+\eta^2]^{\frac12}\times\Big(\frac{2\pi\eta}{e^{2\pi\eta}-1}\Big)^{\frac12}$$

(F désignant la fonction hypergéométrique confluente), et que l'on peut écrire d'autre part:

$$F_0+iG_0=\frac{ie^{-i\rho}}{C_0}\int_0^\infty t^{-i\eta}(t+2i\varrho)^{i\eta}e^{-t}dt.$$

La note contient de très nombreux résultats numériques. *R. Campbell* (Caen).

★**Kahn, Herman. Use of different Monte Carlo sampling techniques.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 146–190. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

The paper may be looked upon as a review of several general procedures that the author and others have considered for reducing sampling variation when doing Monte Carlo type estimation. In some cases these procedures are immediate applications of well known statistical sampling techniques to particular physical problems. Even in these cases the author has given examples which illustrate that the "real" improvements in reducing sampling fluctuations are obtained by utilizing knowledge of the specific underlying physical model that is being studied. The theme of always looking to see if knowledge of the underlying physical process may be used to help reduce sampling variation is not new and this point may be obvious to "experts" in this field, but no doubt this aspect of the paper should be valuable to others.

In the first part of the paper the author uses unrealistic examples (the author describes the examples as trivial) to help the reader gain insight into the various sampling approaches he plans to review. The rest of the paper is concerned with how these procedures may help reduce sampling fluctuations when estimating the values of definite integrals by Monte Carlo Methods.

{The reader should be careful that he does not get the impression that one always does better by biasing his sampling and then reweighting the estimators. While it is true that an experimenter has more control of his samples in a Monte Carlo procedure than in a conventional statistical sampling problem one may still be lead to serious pitfalls using this technique. The reason the examples in the first part of the paper look so convincing is that one already knows the true answer to his problem before one starts sampling.

Since the reviewer does not see much point in listing the names of the various sampling plans found in the paper and since to do anything more extensive is not possible in a review, the reviewer wants to emphasize that this paper justifies careful reading for those interested in Monte Carlo methods since the author has made numerous practical suggestions which warrant consideration. There are a few obvious misprints and a few questionable statements but if the reader excercises a little good judgement he should not be bothered by these details and should find the paper rewarding.} *M. Muller.*

Bouckaert, Louis. Les méthodes de Monte Carlo. Rev. Questions Sci. (5) **17** (1956), 344–359.

This is an expository article on the application of Monte Carlo methods. Buffon's needle problem, a partial differential equation and matrix calculations furnish the illustrations. *H. Teicher* (Lafayette, Ind.).

Bauer, Friedrich L. Beiträge zur Entwicklung numerischer Verfahren für programmgesteuerte Rechenanlagen. I. Quadratisch konvergente Durchführung der Bernoulli-Jacobischen Methode zur Nullstellenbestimmung von Polynomen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 275–303 (1955).

This paper takes the classical method of Bernoulli, as modified by Lagrange, and introduce the idea of calculating modulo the original polynomial $P(x)$. It is shown that the convergence of the corresponding iteration process is quadratic.

The cases of equal maximum modulus roots, and of multiple roots are both discussed carefully.

R. W. Hamming (Murray Hill, N.J.).

Schröder, Johann. Neue Fehlerabschätzungen für verschiedene Iterationsverfahren. Z. Angew. Math. Mech. 36 (1956), 168–181. (English, French and Russian summaries)

"Starting from a theorem concerning the abstract iteration process $u_{n+1}=Tu_n$, proved elsewhere, improved estimations of errors for iteration processes in the case of linear and non-linear equation systems as well as initial and boundary value problems in ordinary differential equations are derived." The author uses a partially ordered space (of vectors) or functions as the set of "distances" between the elements to which he applies the operator T. He is thus able to get sharper error estimates and weaker convergence criteria. He gives results for the Richardson and the Liebmann iteration methods for solving simultaneous systems of linear and non-linear equations.

E. Isaacson (New York, N.Y.).

★ **Butler, James W. Machine sampling from given probability distributions.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 249–264. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

The paper begins by assuming that the problem of how one can generate "random" numbers on a computer is solved. Subject to this assumption several known general machine procedures which involve the use of "uniform random" numbers for generating sample values from prescribed probability distribution are systematically reviewed, several interesting analytical examples are included.

M. Muller (Princeton, N.J.).

★ **Motzkin, Theodore S. Neighbor sets for random walks and difference equations.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 47–51. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

For $n>1$ it is shown that a class of random walks could take place on an n-dimensional lattice (each point of the lattice has a neighbor set of s points, $3\leqq s\leqq 2n$) and that the random walks and the neighbor sets of the lattice can simultaneously satisfy a set of eight requirements which have been specified. Subject to these requirements the density of neighbor sets is examined in terms of equivalence classes of lattice points under the group of n-dimensional translations.

M. Muller (Princeton, N.J.).

★ **Trotter, Hale F.; and Tukey, John W. Conditional Monte Carlo for normal samples.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 64–79. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

In the Introductory Note of the papers presented at the Symposium, A. W. Marshall described this paper in the following terms: "The Trotter and Tukey paper is probably the most exciting paper in the volume." "This paper deserves close study (and needs it) to get the Conditional Monte Carlo trick straight." This reviewer does not care to look upon this paper as a collection of tricks. Unfortunately, since much of the material presented is of a heuristic nature, it is not immediately apparent when the techniques suggested in this paper are valid. However, it should be clear to the reader that the authors were more concerned with getting people interested in their ideas than in justifying the fact that the various techniques they suggested have a valid basis.

The authors begin by very quickly stating what they feel constitutes a Monte Carlo procedure. They then speak of generating samples and weights which match them. They then infer the very strong statement that "any sample can come from any distribution". If the reader interprets the remarks in terms of mean values the statements are more meaningful. For example, considerable use is made of the statement that "x is a random variable which is NID (0, 1)-(normally and independently distributed with mean zero and unit variance) with weight $w(x)$." This should be interpreted as follows: Let x be a random variable, assume (without loss of generality) it has a density function $f(x)$. x is NID (0, 1) with weight $w(x)$ if for "all reasonable real-valued functions ϕ"

$$\int_{-\infty}^{\infty}\phi(x)w(x)f(x)dx=\int_{-\infty}^{\infty}\phi(x)(2\pi)^{-\frac{1}{2}}\exp(-x^2/2)dx.$$

One of the difficulties encountered in reading the paper is that often a random variable may be used in a "weighted sense" without any knowledge about the variable, or the class of "reasonable functions". However, in particular situations this difficulty should not be present.

The use of the general concept, "Conditional Monte Carlo", is not new. For example, in the evaluation of multiple integral it may be convenient to iterate the integral using conditional probability distributions (when this is valid) and to sample subject to conditions on at least one of the variables. This may be followed by a numerical integration. [For example: see the paper reviewed fourth below.]

The present paper goes beyond this point. In the first place the authors have found examples where it is very useful to decompose the integral before considering conditional sampling. For example: Suppose it is desired to evaluate $\Pr[A(z)\geqq a \text{ or } B(z)\geqq b]$. In most reasonable situations this may be written (following the notation of the paper) as follows:

$$\Pr[A(z)\geqq a \text{ or } B(z)\leqq b]=\int_{-\infty}^{a}\Pr[B(z)\geqq b|A(z)=(t)]dF_A(t)+\Pr[A(z)\geqq a],$$

where $\Pr[B(z)\geqq b|A(z)=t]$ signifies the probability that $B(z)\geqq b$ under the condition that $A(z)=t$, and $F_A(t)=\Pr[A(z)=t]$ is the distribution function of $A(z)$. This may be a useful decomposition if $\Pr[A(z)\geqq a]$ can be evaluated analytically or approximated easily. This term may even dominate the first term. Attention is then given to evaluating the first term by a Monte Carlo procedure.

In order to estimate values of the conditional probability appearing in the integrand this will involve sampling subject to conditions such as $A(z)=t$. It is precisely at this point that new and interesting sampling procedures are suggested. (It is at this point that the reviewer feels that a more rigorous presentation would have been more useful, especially for those thinking of future applications.) It may very well happen that the "event" $A(z)=t$ is "rare" but important. The obvious Monte Carlo procedure would force one to throw away most of the samples in order to meet the condition that $A(z)=t$. Instead of "throwing away" observations which do not satisfy this condition each observation is transformed so as to satisfy this condition. This goal is justified

(heuristically) by introducing an additional weight to go with each observation. The weighting must be such that one still obtains an unbiased estimator of the conditional probability. No attention is given concerning the selections of the weights in order to insure that this type of estimator has a "reasonable" variance. In fact, the question of the variance of this procedure is left as an open question; however, they have found excellent success in a particular example. The next step of the procedure would be to estimate the conditional probability for other values of t in order to be able to employ a numerical integration technique to approximate the integral. Having taken one bold step forward the authors proceed to suggest that rather than using a different set of observations for each selected value of t, one could use the same set of observations for all t, each time using a new set of weights. They do not claim that this last procedure should always be used, or that it will always be economical, however they have used it successfully.

M. Muller (Princeton, N.J.).

★**Arnold, Harvey J.; Bucher, Bradley D.; Trotter, Hale F.; and Tukey, John W. Monte Carlo techniques in a complex problem about normal samples.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 80–88. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

This paper applies the techniques developed in the paper reviewed above to the following problem. Let x_1, x_2, x_3, x_4 denote the ordered values of four normal deviates of unit variance having the same mean. Let $G=\max[x_2-x_1, x_3-x_2, x_4-x_3]$, largest gap, let $H=\max[x_3-x_1, x_4-x_2]$, largest 3-stretch, let $J=x_4-x_1$, largest 4-stretch or the range. It is desired to determine J_0 where J_0 is defined by $\Pr[G\geqq 3.17$ or $H\geqq 3.31$ or $J\geqq J_0]=5\%$. Results of numerical computations for this problem are summarized to demonstrate that the sampling techniques developed in the first paper can be useful.

M. Muller (Princeton, N.J.).

★**Marshall, Andrew W. The use of multistage sampling schemes in Monte Carlo computations.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 123–140. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

The technique of Importance sampling is reviewed. That is, the problem is to estimate $\xi=\int_{-\infty}^{\infty} g(x)f(x,\theta)dx$, where $f(x,\theta)$ is a probability density function with θ known. The obvious estimator for a sample size n is $\xi_1=n^{-1}\sum_{i=1}^{n} g(x_i)$. Instead one selects observations from another probability density $h(x)$, (the "importance function), and estimates ξ by $\xi_2=n^{-1}\sum_{i=1}^{n} g(x_i)/h(x_i)$. In general it is not at all obvious how one selects $h(x)$. A judicious selection of $h(x)$ can lead to a significant reduction in the variance of the estimator ξ_2. Some of the drawbacks possible when using this procedure are reviewed, e.g. an estimator of infinite variance is possible. Assuming one has some a priori reason for knowing how to select a parametric class of densities, $h(x,\alpha)$, the author suggests criteria for a two stage sampling procedure which utilizes $n=n_1+n_2$ observations where one takes n_1 observations from $f(x,\theta)$ and n_2 observations from $h(x,\alpha_1)$. A weighted estimator for ξ is suggested. There are numerous ways to select n_1, n_2, and α_1. One possibility for selecting α_1 is to use the n_1 observations in a manner that allows α_0 to be estimated by α_1, where α_0 is supposed to be the value of α yielding the estimator from $h(x,\alpha)$ of smallest variance. There then still remains the problem of determining the size of n_1 so that the combined weighted estimator based on n observations has some "optimum" properties. A general multi-stage sampling scheme such as this will still possess the undesirable properties that single-stage "Importance" sampling procedures possess. However, in specific situations the multi-stage approach may be useful. Many unanswered questions still remain. An asymptotic minimum expected loss sequential procedure for estimating ξ is suggested; as pointed out by the author this is very similar to Wald's approach for sequential point estimation [Proc. 2nd Berkeley Symposium Math. Statist., 1950, Univ. of California Press 1951; pp. 1–11; MR 13, 367].

M. Muller.

★**Walsh, John E. Questionable usefulness of variance for measuring estimate accuracy in Monte Carlo importance sampling problems.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 141–144. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

Since it is possible to have estimates of infinite variance when using importance sampling procedures the author suggests using measures other than the variance as a criterion of precision when selecting the new sampling density function $h(x)$ [see the preceding review]. A particular example where the variance is infinite but the mean difference is finite is given.

M. Muller.

Bjerhammar, Arne. Triangular matrices for adjustment of triangular networks. Kungl. Tekn. Högsk. Handl. Stockholm no. **105** (1956), 82 pp.

In the solution of geodetic normal equations by the method of Gauss and Doolittle or that of Cholesky, the symmetric matrix of coefficients of the normal equations is factored into the product of two triangular matrices. Bjerhammar proposes the use of such triangular matrices or their inverses in the reduction of those normal equations arising out of the simple angle condition equations in the adjustment of a triangulation network. This technique appears more flexible than the familiar methods developed by Boltz, which utilize square inverse matrices.

The author has tabulated the rows of triangular matrices and of triangular reciprocal matrices for each of several canonical forms, determined to within a permutation of the elements by the number and incidence of the triangles in the chains and connecting figures. The order of occurrence of these rows is listed for about 150 different configurations, and instructions for the combination of chains are given. These tables should provide a convenient aid to computation for those solving adjustments by condition equations on desk calculators or by hand computations. People having access to large-scale computing machines and those preferring the variation of coordinates method of adjustment may find this of less interest than the author's earlier works on this subject.

I. Gaskill (Washington, D.C.).

Gotthardt, E. Zur zweistufigen vermittelnden Ausgleichung. Z. Vermessungswesen **81** (1956), 241–244.

Kennemann, Wolfgang. Anwendungsbereiche der Näherungsformeln für die goniometrischen Funktionen kleiner Winkel. Allg. Vermessgs.-Nachr. **1956**, 218–223.

The author computes the limits, in the decimal division of angles, for which a given accuracy in the values of $\sin\alpha$, $\tan\alpha$, $\log\sin\alpha$ and $\log\tan\alpha$ can be obtained by

approximating formulas. In particular the rules of Maskelyn are treated. Finally the inverse problem is discussed. *E. M. Bruins* (Amsterdam).

Fraeys de Veubeke, B. Matrices de projection et techniques d'itération. Ann. Soc. Sci. Bruxelles. Sér. I. **70** (1956), 37–61.

Soit M une matrice symétrique définie positive, a et b deux colonnes. La matrice

$$Q=E-\frac{ab'M}{b'Ma}$$

s'interprète comme une projection parallèlement à a sur l'hyperplan orthogonal à b. L'auteur étudie ces matrices projections et ceux de leurs produits qui sont des projections multiples. Ces notions permettent de présenter sous une forme plus intuitive diverses méthodes de recherches des valeurs propres. *J. Kuntzmann.*

Wilkinson, J. H. The use of iterative methods for finding the latent roots and vectors of matrices. Math. Tables Aids Comput. **9** (1955), 184–191.

Report on solving algebraic eigenvalue problems on the pilot ACE computer. The iteration methods used for this purpose have been described by the author recently [Proc. Cambridge Philos. Soc. **50** (1954), 536–566; MR **16**, 178]. The routine is extremely simple. Two sequences y_s and z_s of iterated vectors are computed by

$$z_s=(A-pI)y_s,$$

$$y_{s+1}=z_s/(\text{element of } z_s \text{ of maximum modulus}),$$

A being the given matrix and p an appropriate chosen number. The vectors z_s tend to a latent vector. Afterwards a matrix of order one less is formed and treated in the same way. Aitken's Δ^2-process is used to improve convergence.

The experience on the pilot ACE with several hundred matrices shows that even higher latent vectors are obtained with high accuracy. The author treated some examples characterised by very close eigenvalues and proves that iteration used in a flexible manner is much better than E. Bodewig believes in a recent paper [same journal **8** (1954), 237–240; MR **16**, 288]. *E. Stiefel.*

Kovalev, I. F.; and Mayanc, L. S. A new method of calculation of derivatives with respect to parameters of the roots of secular equations. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 175–178. (Russian)

The method used by the authors is based on the property that every root of a secular equation (1) $|W-\lambda E|=0$ can be considered as an implicit function of parameters t_j, $j=1, 2, \cdots, m$, on which the elements of the matrix W depend. Therefore, for every root of the equation (1) one can write $\phi(\lambda_i, t_j)=|W-\lambda_i E|=0$. Assuming that λ_1 is a simple root, the authors show that

$$\partial\lambda_i/\partial t_j=\mathrm{Sp}(\partial W/\partial t_j cA^{(i)})/\mathrm{Sp}(cA^{(i)}),$$

where c is arbitrary and the matrix $A^{(i)}$ satisfies the equation $(W-\lambda_i E)A^{(i)}=|W-\lambda_i E|E$. Next they give a procedure for overcoming the difficulty that arises from the fact that the matrix $W-\lambda_i E$ has no inverse.

S. Kulik (Columbia, S.C.).

Bauer, Friedrich L. Das Verfahren der abgekürzten Iteration für algebraische Eigenwertprobleme, insbesondere zur Nullstellenbestimmung eines Polynoms. Z. Angew. Math. Phys. **7** (1956), 17–32.

Given a polynomial of degree n, one complete step of the proposed "abbreviated iteration" consists of n single steps of the usual Bernoullian iteration as modified by Lagrange.

Except for scale factor changes, the entire iteration may be set up without divisions; it therefore is well adapted to finding the zeros of polynomials with complex coefficients.

The iteration process gives not only the zero to be determined, but also the residual polynomial. In the case of two or more dominating roots of equal modulus the iteration process does not converge, but by successive processes analogous to the "cross-multiplication" of Routh a derived sequence may be formed which does converge.

The original polynomial may be used without any change for the determination of each zero, and there is propagation of roundoff error. Each iteration is self-correcting and may easily be checked. *R. W. Hamming.*

Gaier, D. Über die Konvergenz des Adamsschen Extrapolationsverfahrens. Z. Angew. Math. Mech. **36** (1956), 230.

The author proves that the Adams method of numerical solution of differential equations converges under the usual conditions. For the maximum error he derives a more sensitive formula which depends on the step size. *J. A. Ward* (Holloman, N.M.).

Krasil'nikov, V. A.; and Obuhov, A. M. On propagation of waves in a medium with random inhomogeneities in the coefficient of refraction. Akust. Ž. **2** (1956), 107–112. (Russian)

This paper is a very concise survey of methods of approximate solution of the wave equation $\phi_{tt}=c^2\nabla^2\phi$ when the velocity c can be written in the form $c=c_0(1+\mu)$, μ being a small parameter depending upon position but not upon time. Such a condition is often encountered when turbulence is present in atmospheric acoustics and optics, underwater acoustics, and radio wave propagation in the troposphere. The methods of perturbation, ray theory in both real and complex form, correlation functions, structure functions, and combinations of these are touched upon with a brief indication of the domain of applicability of each. Nearly all the references are to works of the authors. *R. N. Goss* (San Diego, Calif.).

See also: Helman, p. 125; Douglas, p. 129; Chakrabarti, p. 134; Urmaev, p. 147; Bjerhammar, p. 153; Falk, p. 161; Bland, p. 164; Betz, p. 168; Brehovskih, p. 169.

Graphical Methods, Nomography

Cunningham, W. J. Graphical solution of certain nonlinear differential-difference equations. J. Franklin Inst. **261** (1956), 621–629.

The equations studied by the author are all of the the general type

$$\dot{x}(t)=f[x(t), x(t-\tau)],$$

where τ is constant. In order to specify a solution uniquely we must know its values for an entire interval $-\tau<t<0$. In this paper it is shown how to construct the curve in the (u, v) plane for the parametric equations $u=x(t)$, $v=\dot{x}(t)$. Two examples of practical interest are worked in detail. *W. E. Milne* (Corvallis, Ore.).

See also: Štykan, p. 135; Moskvitin, p. 163.

Tables

★**Faddeeva, V. N.; and Terent'ev, N. M. Tables of values of the probability integral {the function**

$$w(z)=e^{-z^2}(1+2i\pi^{-\frac{1}{2}}\int_0^z e^{t^2}dt)\}$$

of complex argument. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 32 pp.

A translation of the article reviewed in MR **16**, 960.

See also: Bjerhammar, p. 153; Raff, p. 160.

Machines

Vitenberg, I. M.; and Gluzberg, E. A. The solution of algebraic, transcendental and integral equations by means of analog computers. Avtomat. i Telemeh. **17** (1956), 590–600, appendix to no. 7, 2. (Russian. English summary)

The authors propose to solve the equations by minimizing the sum of absolute values of residuals, and for the minimization suggest auxiliary devices for which circuit diagrams are given. *A. S. Householder.*

Ercoli, Paolo; Sacerdoti, Giorgio; e Vacca, Roberto. La tecnica britannica nel campo delle calcolatrici elettroniche automatiche alla luce del recente Congresso di Londra. Ricerca Sci. **26** (1956), 2321–2339.

An account of a series of visits to the computing laboratories of England. Very brief descriptions are given of a half-dozen large scale computers. *D. H. Lehmer.*

Ledley, Robert S. Digital computational methods in symbolic logic, with examples in biochemistry. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 498–511; errata, 796.

In two previous papers [Nat. Bur. Standards Rep. 3363 (1954); J. Operations Res. Soc. Amer. **2** (1954), 249–274; MR **16**, 175] the author proposed a technique useful for the mechanical solution of problems in Boolean algebra. The principle of this technique is that an element in a free Boolean algebra with m generators can be represented as a binary number with 2^m digits, these digits being the coefficients in a "development" (in the sense of Schröder) in which the atoms are arranged in some standard order. The present paper contains some improvements in his methods. The procedures for solving Boolean equations are presented in terms of Boolean matrices, and consideration is given to cases where the generators are subject to conditions ("constraints"). At the end the author discusses two interesting and nontrivial examples from biochemistry. It now seems clear that the author's methods save some labor in the manipulation of complex equations; but his treatment is marred by a propensity to make simple things appear difficult, and by peculiarities of notation (such as numbering his digits from right to left and his rows from the bottom upwards) which conceal the relationship of his methods to those which are more familiar. *H. B. Curry.*

See also: Davis, p. 103; Shannon, p. 103; McCarthy, p. 103; de Leeuw, Moore, Shannon and Shapiro, p. 104; Postley, p. 107; Butler, p. 152; Wilkinson, p. 154; Falk, p. 161; Craven, p. 168.

PROBABILITY

★**Kolmogorov, A. N. Foundations of the theory of probability. Second English edition.** Translation edited by Nathan Morrison, with an added bibliography by A. T. Bharucha-Reid. Chelsea Publishing Co., New York, 1956. viii+84 pp. $2.50.

In an editor's note it is stated that in preparation of this English translation the original German monograph, which appeared in Ergebnisse der Mathematik in 1933, and also a Russian translation by G. M. Bavli published in 1936 have been used.

Komatu, Yûsaku. Elementary inequalities for Mills' ratio. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1955), 69–70.

An elementary proof of the inequality

$$\frac{2}{\sqrt{(x^2+4)}+x}<e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}v^2}dt<\frac{2}{\sqrt{(x^2+2)}+x}$$

Z. W. Birnbaum (Seattle, Wash.).

Lal, D. N. A note on a form of Tchebycheff's inequality for two or more variables. Sankhyā **15** (1955), 317–320.

Let (X_1, X_2) be a two-dimensional random variable with both expectations zero, variances σ_1^2, σ_2^2, and correlation coefficient r. The author derives the inequality

$$\Pr\{|X_1|<k_1\sigma_1,\ |X_2|<k_2\sigma_2\}\geqq 1-\left(k_1^{-2}+k_2^{-2}+\frac{\sqrt{[(k_1^{-2}+k_2^{-2})-4r^2k_1^{-2}k_2^{-2}]}}{2k_1^{-2}k_2^{-2}}\right).$$

This result is extended to n-dimensional random variables. *Z. W. Birnbaum* (Seattle, Wash.).

Darling, D. A.; and Siegert, A. J. F. On the distribution of certain functionals of Markoff chains and processes. Proc. Nat. Acad. Sci. U.S.A. **42**(1956), 525–529.

Es sei $(x_n;\ n=0,\ 1,\ \cdots)$ eine Markoffsche Kette mit Werten in einer Menge Ω, konstantem $x_0=x$ und stationären Übergangswahrscheinlichkeiten

$$P_k(x, E)=\Pr\{x_{n+k}\in E|x_k=x\},$$

$E\in\mathfrak{F}$, wobei $\mathfrak{F}$ eine sigma-Mengenalgebra in Ω bilde. Ferner sei $V(y)$ eine in Ω erklärte $\mathfrak{F}$-meßbare reelle Funktion.Zur Bestimmung der Verteilung von $U_n=V(x_0)+\cdots+V(x_n)$ wird die Funktion

$$R(x, E)=\sum_{n=0}^{\infty}\mathrm{E}\{e^{i\xi U_n}|x_0=x,\ x_n\in E\}P_n(x, E)z^n$$

mit $0\leqq z<1$ und reellem ξ gebildet und gezeigt, daß sie die beiden folgenden Integralgleichungen erfüllt und dadurch eindeutig bestimmt ist:

$$R(x, E)=P_z(x, E)+\int(1-e^{-i\xi V(y)})P_z(x, dy)(Ry, E)$$

und die durch Vertauschung von R und P_z unter dem Integralzeichen entstehende Gleichung, wobei $P_z(x, E)=\sum_{n=0}^{\infty}P_n(x, E)z^n$. Entsprechendes trifft auf kontinuierliche Markoffsche Prozesse mit stationären Übergangswahrscheinlichkeiten zu. Im Fall des euklidischen Raumes $\Omega=E^n$ und eines Prozesses, der eine Dichte hat, werden

heuristische Betrachtungen über eine partielle Differentialgleichung für die Dichte von R angestellt und ein Beispiel behandelt. (In der Formel (2.2) des Originals fehlen Faktoren $P(x,E,t)dt$.). *K. Krickeberg.*

Baxter, Glen. Wiener process distributions of the "arc-sine law" type. Proc. Amer. Math. Soc. **7** (1956), 738–741.

The arc sine law for the Wiener process states that [Erdös and Kac, Bull. Amer. Math. Soc. **53** (1947), 1011–1020; MR **9**, 292]

$$P\{m(s \in [0, T]: x(s)>0) \leqq t\} = \frac{2}{\pi} \arcsin\left(\frac{t}{T}\right)^{1/2}.$$

The author generalises this result by calculating the distribution function of $m(s \in [0, T]: x(s)>as)$ for every $a \geqq 0$. Among others he also obtains the following result: Denote by $\varphi(a, T)$ the last zero of $x(s)-as$ before $s=T$. Then for $u>0$

$$(1) \qquad P\{\varphi(a, \infty) \leqq u\} = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{au^{\frac{1}{2}}} e^{-\xi^2/2} d\xi.$$

The author remarks that (1) immediately implies that with probability one the s-set for which $x(0)>as$ is bounded ($a>1$), i.e. (1) implies the strong law of large numbers. *P. Erdös.*

Takano, Kinsaku. Multidimensional central limit criterion in the case of bounded variances. Ann. Inst. Statist. Math., Tokyo **7** (1956), 81–93.

Takano, Kinsaku. Central convergence criterion in the multidimensional case. Ann. Inst. Statist. Math., Tokyo **7** (1956), 95–102.

Takano gives in these articles criterions for convergence of the distribution function for random vectors towards an infinitely divisible law with log of characteristic function of the form

$$\psi(t) = \log E e^{it'x} =$$

$$iat - \tfrac{1}{2} t' \sigma t + \int_{R_p} \left(e^{it'x} - 1 - \frac{it'x}{1+|x|^2} \right) dv \quad (t \in R_p),$$

where

$$\int_{|x|<1} |x|^2 dv < \infty, \quad \int_{|x| \leq 1} dv < \infty.$$

The first paper contains three theorems which are generalizations to the p-dimensional space of similar theorems formulated for the one dimensional space in a work by Gnedenko and Kolmogorov [Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949; MR **12**, 839; **16**, 52]. The author specializes these theorems to special criterions for convergence of the law for sums of p-dimensional vectors towards a normal p-dimensional law and for convergence towards convolutions of a finite number of multidimensional Poisson laws or Poisson laws and a normal law. In the second paper Takano gives equivalent criterions for the same purpose. *L. Törnqvist* (Helsingfors).

Dugué, D. Sur les théorèmes limites du calcul des probabilités. Rev. Inst. Internat. Statist. **23** (1955), 29–35.

This paper gives necessary and sufficient conditions connecting the behavior of the real part of the characteristic function around zero and the behaviour of the distribution function around infinity.

Conditions for compactness, existence of a derivative are given as well as the construction of a characteristic function whose derivatives at the origin are different. *Author's summary.*

Spitzer, Frank. A combinatorial lemma and its application to probability theory. Trans. Amer. Math. Soc. **82** (1956), 323–339.

Let $X_1, X_2, \cdots$ be independent identically distributed random variables, and S_k their partial sums. Let $\phi_n(\lambda)$ be the characteristic function of $\max_{1 \leq k \leq n} S_k^+$ and $\psi_n(\lambda)$ the characteristic function of S_n^+. The author proves that

$$\sum_{n=0}^{\infty} \phi_n(\lambda) t^n = \exp\left[\sum_{n=1}^{\infty} \frac{\psi_n(\lambda)}{n} t^n \right].$$

This result generalizes a result of Kac and Hunt [Kac, Duke Math. J. **21** (1954), 501–509; MR **16**, 31] which stated that $E \max_{1 \leq k \leq n} S_k^+ = \sum_{k=1}^{n} k^{-1} E S_k^+$ and a result of E. S. Andersen [Math. Scand. **2** (1954), 195–223; MR **16**, 839] which stated that

$$\sum_{n=0}^{\infty} \Pr\left\{ \bigcap_{k=1}^{n} [S_k>0] \right\} t^n = \exp\left\{ \sum_{k=1}^{\infty} \frac{t^k}{k} \Pr[S_k>0] \right\}.$$

These last two results hold for the slightly more general case of dependent variables having joint distribution invariant under permutations of the indicies. Using the above result the author obtains new results on the limiting behavior of $\max_{1 \leq k \leq n} S_k^+$, N_n=number of the first n S_k which are positive, and T_n= the smallest $i \geqq 0$ for which $S_i = \max(0, S_1, \cdots, S_n)$. For example, let $a_k = \Pr[S_k>0]$. If $\sum_{k=1}^{\infty} a_k/k < \infty$, then $\max_{k \geq 1} S_k^+$ is finite with probability one and has the infinitely divisible characteristic function $\prod_{k=1}^{\infty} \exp\{(\psi_k(\lambda)-1)/k\}$. If $\Pr[X_i=0] \neq 1$, then $\limsup_{n \to \infty} S_n = -\infty$ with probability one. Also $N_n \to N < \infty$ and $T_n \to T < \infty$ with probability one and $Et^N = Et^T = \exp\{\sum_1^{\infty} a_k(t^k-1)/k\}$ for $|t| \leqq 1$. If $\sum_1^{\infty} a_k/k = \infty$, then $\sup_{k \geq 1} S_k^+ = \infty$, and N_n and $T_n \to \infty$ with probability one. The author also generalizes the result of Andersen [loc. cit.] which states that if a_k converges to α, then N_n/n converges in distribution to an arc-sin law with parameter α. The author proves that it is sufficient that a_k be $(C, 1)$ summable to α. *J. L. Snell* (Hanover, N.H.).

Kesten, H.; and Runnenburg, J. Th. Some elementary proofs in renewal theory with applications to waiting times. Math. Centrum Amsterdam. Statist. Afdeling Rep. S **203** (1956), 16 pp.

Elementary proofs are given for the following results in renewal theory. Let $\{x_n, n=1, 2, \cdots\}$ be independent, positive, nonlattice random variables with the same cdf F, and define $s_k = x_1 + \cdots + x_k$, $a(t) = \max_{s_n \leq t} s_k$, $b(t) = \min_{s_k > t} s_k$, $\mu_i = Ex_1^i$, $U(t) = \sum \Pr(s_k \leqq t)$. Then for every bounded function $k(t)$ vanishing for $t<0$, nonincreasing for $t \geqq 0$ and with

$$\int_0^{\infty} k(t)dt < \infty, \quad \int_0^{\infty} k(t-x)dU(x) \to \int_0^{\infty} k(x)dx/\mu_1 \text{ as } t \to \infty.$$

If $\mu_1 < \infty$, $b(t)-t$ has a limiting distribution as $t \to \infty$ with density $(1-F(w))/\mu_1$, $w \geqq 0$, and $b(t)-a(t)$ has a limiting distribution as $t \to \infty$ with cdf $\int_0^w xdF/\mu_1$, $w \geqq 0$. If $\mu_2 < \infty$, $U(t) = (t/\mu_1) + (\mu_2/2\mu_1^2) - 1 + o(1)$ as $t \to \infty$. *D. Blackwell* (Berkeley, Calif.).

Onoyama, Takuji. Regular random functions and linear translatable stochastic functional equations. Mem. Fac. Sci. Kyusyu Univ. Ser. A. **9** (1956), 135–158.

A real-valued stochastic process $z(t, \omega)$ depending upon

a continuous time parameter t and a random parameter ω is said to be a regular random function (r.r.f.), if the variance $E_\omega\{|z(t,\omega)-E_\omega\{z(t,\omega)\}|^2\}$ exists and

$$E_\omega\{|z(t+h,\omega)-z(t,\omega)|^2\}\to 0$$

as $h\to 0$ for all t. This paper discusses the non-homogeneous linear translatable stochastic functional equation

$$\Lambda_t f(t,\omega)=g(t,\omega),$$

where $g(t,\omega)$ is a given r.r.f., and the operator Λ is a linear translatable operation in the sense of T. Kitagawa. The author establishes a expansion theorem which correspond to Weyl-Stone-Titchmarsh-Kodaira's eigenfunction expansion formula associated with a second order differential equation. By making use of this expansion theorem the author gives two representation formulas of the solutions of the functional equation, under either of the two conditions (i) and (ii). The condition (i) means that the generating function of the linear translatable operator $G(z)=e^{-zt}\Lambda_t e^{zt}$ has no imaginary zero-point, while the condition (ii) means some detailed restrictions as to the set of the pure imaginary roots.

Finally the author discusses the estimation of the mean function of r.r.f. $E_\omega\{X(t,\omega)\}=m(t)$, as an application of the expansion formula given above. *T. Kitagawa.*

Špaček, Antonín. Zufällige Gleichungen. Czechoslovak Math. J. **5(80)** (1955), 462–466. (Russian summary)

A "generalized random variable" is defined to be a measurable transformation from a measurable space M to a complete metric space X. A transformation $T(u,x)$ which carries points $u\in M$, $x\in X$ into is then a stochastic transformation if for each x it is a generalized random variable. If in addition T satisfies a Lipschitz condition, $\varrho(T(u,x),T(u,y))\leqq c\varrho(x,y)$, where ϱ is the metric and $c<1$, then for each $u\in M$, T has a unique fixed point $f(u)\in X$. With further restrictions on M, X and T it is shown that f is a generalized random variable. As an application the author treats a Fredholm integral equation with a stochastic kernel. *G. Hufford.*

★ **Iaglom, A. M. Effective solution of linear approximation problems for processes with random, stationary n-th increments.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp. A translation of the article reviewed in MR **16**, 494.

Theodorescu, Radu. Chaînes continues à liaisons complètes de multiplicité p. Com. Acad. R. P. Romîne **6** (1956), 253–257. (Romanian. Russian and French summaries)

L'auteur introduit un système d'équations dont il démontre que la solution (unique) représente une probabilité et vérifie une équation fonctionnelle où entrent les probabilités d'une chaîne de Markoff. *O. Bottema* (Delft).

Clarke, A. B. A waiting line process of Markov type. Ann. Math. Statist. **27** (1956), 452–459.

The author studies the Markov process associated with the Kolmogorov equations,

$$p_{ij}'=\lambda p_{i,j-1}-(\lambda+\mu)p_{ij}+\mu p_{i,j+1}\ (j\geqq 1),$$
$$p_{i0}'=-\lambda p_{i0}+\mu p_{i1},$$

where λ and μ are functions of the time. When μ is a constant we obtain the "queue" (waiting line) process corresponding to a single server, a negative-exponentially distributed service-time and a Poisson input of variable intensity. The author writes $\tau=\int_0 \mu(s)ds$, $\varrho(\tau)=\lambda(t)/\mu(t)$,

$$R(\tau)=\frac{1}{\tau}\int_0^\tau \varrho(\sigma)d\sigma$$

and $Q_{ij}=e^{\tau[1+R(\tau)]}p_{ij}$; he then introduces the novel generating function, $Q_i(z,\tau)=\sum Q_{ij}(\tau)^j/j!$, and shows that it satisfies the telegraphers' equation,

$$(*)\qquad \frac{\partial^2 Q_i}{\partial\tau\partial z}=\varrho(\tau)Q_i,$$

together with centrain boundary conditions. The classical Riemann method of solving (*) then leads to formulae for the transition probabilities $p_{ij}(t)$ in terms of a function $f_i(\tau)$ which is to be determined as the solution to a Volterra type integral equation.

In the second part of the paper it is supposed that λ and μ are both constants (or, more generally, that ϱ is constant). For this special case the author carries out his programme in full and obtains the $p_{ij}(t)$ as sums of infinite series of Bessel functions. Essentially equivalent formulae were obtained earlier by W. Ledermann and G. E. H. Reuter [Philos. Trans. Roy. Soc. London. Ser. A. **246** (1954), 321–369, pp. 365–366; MR **15**, 625], and yet another approach to the same problem was given recently by N. T. J. Bailey [J. Roy. Statist. Soc. Ser. B. **16** (1954), 288–291; MR **16**, 1035].

In an exceedingly useful final section the author studies the growth in time of the means and variances of the queue size and waiting time when $\varrho\geqq 1$ (so that there is no limiting equilibrium). The only earlier paper on the non-stationary case which gives detailed information of this kind is that of O. Volberg [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **24** (1939), 657–661; MR **1**, 345] and the author's contribution to this problem should find many immediate applications. *D. G. Kendall* (Oxford).

Wishart, David M. G. A queueing system with χ^2 service-time distribution. Ann. Math. Statist. **27** (1956), 768–779.

The author considers a single-server queue with general independent service times. The intervals which separate the customers' arrival instants are assumed to be independent samples from a scale-modified χ^2 distribution with an even number, $2k$, of degrees of freedom. It is possible, therefore, to regard each of these intervals as the sum of k independent samples from a certain negative-exponential distribution. This fact enables the author to evaluate the ergodic properties of his model by making use of the approach developed by D. G. Kendall [same Ann. **24** (1953), 338–354; MR **15**, 44]. *W. L. Smith.*

Jackson, R. R. P. Random queueing processes with phase-type service. J. Roy. Statist. Soc. Ser. B. **18** (1956), 129–132.

The system considered is one in which customers are served at each of a number of phases (counters) arranged in series; every customer commences service at the first phase and continues through the system, receiving attention at all of the others until he is discharged after the last phase. At each phase there are a number of identical servers. The distribution of the service time at each phase and of the interval between successive arrivals is assumed negative exponential. Let $P(n_1,n_2,\cdots,n_k,t)$ be the probability that at time t, there are n_1 customers in the first phase of service, n_2 customers in the second

phase, and so on. The author assumes that

$$P(n_1, n_2, \cdots, n_k, t)$$

approaches a limit as $t \to \infty$, and finds this limit.

J. Wolfowitz (Ithaca, N.Y.).

Pugačev, V. S. A general condition for minimizing the mean square error for a dynamical system. Avtomat. i Telemeh. **17** (1956), 289–295; appendix to no. 4, 1–2. (Russian. English summary)

If $Y(z)$, (the "signal"), and $X(t)$, (the "noise") are suitable complex-valued stochastic processes, and R a linear space of operators on $X(t)$, (the operators themselves may be nonlinear), a necessary and sufficient condition that $AX(t)$ be a least-squares estimate of $Y(z)$ is given by

(*) $\operatorname{Re} E\{[Y(z)-AX(t)][\overline{AX(t)}-\overline{BX(t)}]\}=0$, all $B \in R$.

Specializing R in different ways formally transforms (*) into relations obtained by numerous workers in the field of filtering theory. In this way one obtains conditions of Wiener, Zadeh and Ragazzini, Booton, Davis, and others.

E. Reich (Minneapolis, Minn.).

Pugačev, V. S. Application of canonical expansions of random functions to problem of determining optimum linear system. Avtomat. i Telemeh. **17** (1956), 489–499, appendix to no. 6, 1–2. (Russian. English summary)

Let $\{X(t)\}$ be a stochastic process, with covariance function value $K(t, s)$. A sequence of (not uniquely determined) linear functionals, $\{\Omega^{(\nu)}\}$ is found, for which

$$\overline{\Omega_t^{(\nu)}}[\Omega_s^{(\mu)}K(t, s)]=\Delta_\mu\delta_{\mu\nu},$$

and then the process is written in the form $X(t)=M\{X(t)\}+\sum V_\mu u_\mu(t)$, where $u_\mu\Delta_\mu$ is the function in the above bracket, and where $V_\mu=\overline{\Omega^{(\mu)}}[X(t)-M\{X(t)\}]$. Let $f_1, \cdots, f_N, g_1, \cdots, g_N$ be specified functions, and let $U_1, \cdots, U_N$ be random variables with zero expectations, orthogonal to the $X(t)$ process as well as to a given $Y(z)$ process. The author gives a formal solution of the problem of best linear least squares approximation to $Y(z)+\sum U_r g_r(z)$ by means of the family of random variables $\hat{X}(t)=X(t)+\sum U_r f_r(t)$. The solution has the form $[\sum x_\nu(z)\overline{\Omega^{(\nu)}}]\hat{X}(t)$. These results generalize earlier results by the author [Sb. Naučnyh Trudov VVIA Žukov. no. 1 (1954); see also the paper reviewed above]. *J. L. Doob.*

Pugačev, V. S. Possible general solution of the problem of determining optimum dynamical system. Avtomat. i Telemeh. **17** (1956), 585–589, appendix to no. 7, 1–2. (Russian. English summary)

In the problem of the preceding review, if $N=0$, and if the random variables involved are jointly normal, in which case it is well-known that the best not necessarily linear operator providing the solution is actually linear, and is the appropriate conditional expectation, the solution is calculated as a conditional expectation.

J. L. Doob (Geneva).

See also: de Leeuw, Moore, Shannon and Shapiro, p. 104; Motzkin, p. 152; Faddeeva and Terent'ev, p. 155; Weiss, p. 158; Hájek, p. 159; Rybarz, p. 160; Benson, p. 180.

STATISTICS

Wold, Herman. Causal inference from observational data, a review of ends and means. J. Roy. Statist. Soc. Ser. A. **119** (1956), 28–50, discussion 51–61.

A broad review of the possibilities and deficiencies of present-day statistical method, particularly in the area indicated by the title. The author divides the field of applied statistics into four parts by the dichotomies, descriptive v. explanatory treatments, experimental v. observational data, and examines questions of varying degrees of generality in terms of this classification: e.g., model-building, estimation and hypothesis-testing with observational data, causal patterns, regression analysis, and models of consumer demand and migration. It is the topic "simultaneous v. recursive regression models" that receives most attention in the ensuing discussion.

P. Whittle (Wellington).

Weiss, Lionel. A certain class of solutions to a moment problem. Ann. Math. Statist. **27** (1956), 851–853.

Let $\mu_0=1, \mu_2, \cdots, \mu_n$ be a given set of real numbers such that there exists at least one density function $f(x)$ on $[-1, 1]$ with

$$\int_{-1}^{1} x^i f(x)dx=\mu_i \ (i=0, \cdots, n), \tag{1}$$

$$\int_{-1}^{1} f^2(x)dx=I(f)<\infty, \tag{2}$$

$$f(x)\geqq 0 \text{ for } -1\leqq x\leqq 1. \tag{3}$$

The author proves that in the class $\{f\}$ of functions satisfying (1), (2) and (3) there exists a function $g(x)$ such that

$$\int_{-1}^{1} g^2(x)dx=\inf_{f\in\{f\}}\int_{-1}^{1} f^2(x)dx$$

and that the function $g(x)$ is unique (save for sets of measure zero). He further shows that $g(x)$ is, almost everywhere on $(-1, 1)$, equal to a certain polynomial $P(x)$ of degree at most n, wherever $P(x)$ is non-negative, and is equal to zero elsewhere. A generalization is indicated; a bounded set S of positive measure replaces $[-1, 1]$, $h_i(x) \in L_2$ replace x^i and the extremal function $g(x)$ is a certain linear combination of the $h_i(x)$ (or zero where that linear combination is negative).

L. E. Moses (Stanford, Calif.).

Matusita, Kameo; and Akaike, Hirotugu. Decision rules, based on the distance, for the problems of independence, invariance and two samples. Ann. Inst. Statist. Math., Tokyo **7** (1956), 67–80.

Matusita, Kameo; and Motoo, Minoru. On the fundamental theorem for the decision rule based on distance ‖ ‖. Ann. Inst. Statist. Math., Tokyo **7** (1956), 137–142.

Doornbos, R. Significance of the smallest of a set of estimated normal variances. Statistica, Neerlandica **10** (1956), 117–126. (Dutch summary)

This is an early version of a paper by Doornbos and Prins [Math. Centrum Amsterdam. Statist. Afdeling Rep. S 187 (VP 4) (1956); MR 17, 641]. *I. R. Savage.*

Kołniewska, I. Comparison of the efficiency of drawing lots with and without returning them, when the variance of the general population is unknown. Zastos. Mat. 2 (1955), 297–304. (Polish. Russian and English summaries)

Let Π be a population of N elements, with mean μ and variance σ^2. A sample S is obtained by drawings with replacements, and a sample S^* without replacements, both of size n. Let M_2 denote the usual unbiased estimate of σ^2 when based on S, and M_2^* when based on S^*, in the case of μ not known; let α_2 and α_2^* denote the estimate $\sum_{i=1}^n (X_i-\mu)^2/n$ of σ^2, based on S or S^*; respectively, in the case of μ known. The author proves:

$$\mathrm{Var}(\alpha_2^*)<\mathrm{Var}(M_2^*)<\mathrm{Var}(M_2) \text{ and}$$

$$\mathrm{Var}(\alpha_2^*)<\mathrm{Var}(\alpha_2)<\mathrm{Var}(M_2),$$

and gives a necessary and sufficient condition for the inequality $\mathrm{Var}(M_2^*)<\mathrm{Var}(\alpha_2)$. *Z. W. Birnbaum.*

Kimball, Bradford F. The bias in certain estimates of the parameters of the extreme-value distribution. Ann. Math. Statist. 27 (1956), 758–767.

The paper deals almost entirely with computing the bias of a new (and simple to compute) estimate of the reciprocal of the scale parameter α in the Type I extreme value distribution described by the c.d.f.

$$\Phi(x, \alpha, u)= \exp\{-\exp[-\alpha(x-u)]\}.$$

The new estimate of α is suggested by its maximum likelihood estimate (which is difficult to compute explicitly). For a sample of size n, where the observations are ordered: $x_1\leqq x_2\leqq\cdots\leqq x_n$, the estimate $\hat{\beta}$ is given by $\hat{\beta}=\sum_{j=1}^n c_j x_j$, where $c_j=(1-\sum_{i=j}^n 1/i)n^{-1}$. Formulae for the constants b_n which are such that $E(b_n\hat{\beta})=1/\alpha$ are worked out for any size sample n and tabulated for $n=2(1)112$. The question of estimating the parameter u, when α is known, is treated briefly. Formulae are given for the bias and variance of this estimator. Results for Type I extreme value distributions can be extended readily to those of Types II and III. *B. Epstein.*

Seal, K. C. On minimum variance among certain linear functions of order statistics. Ann. Math. Statist. 27 (1956), 854–855.

Let one observation be drawn at random from each of n normal populations $N(\mu_i, 1)$, $i=1, 2, \cdots, n$. Let $x_1\leqq x_2\leqq\cdots\leqq x_n$ be the observations arranged in order of size and let the corresponding n random variables be X_i, $i=1, 2, \cdots, n$. It is proved that $\mathrm{Var}(\sum_{i=1}^n c_i X_i)$, where $\sum_{i=1}^n c_i=1$, is minimized by choosing $c_i=1/n$, $i=1, 2, \cdots, n$. *B. Epstein* (Detroit, Mich.).

Chapman, D. G.; and Junge, C. O., Jr. The estimation of the size of a stratified animal population. Ann. Math. Statist. 27 (1956), 375–389.

In some animal populations, there may be a natural stratification by location and partial migration between strata. The total size N of such a population is to be estimated, when a tagging procedure is used in each stratum. Alternative probability models and several estimates of N are discussed. The author defines consistency as convergence in probability with the additional requirement that the ratio of stratum sample to population sizes and the ratio of certain other random variables behave in a particular fashion, and considers consistency of the estimates. The asymptotic variance of the estimates of N and an estimate of migration are also given. *I. Olkin* (East Lansing, Mich.).

Dwass, Meyer. The large-sample power of rank order tests in the two-sample problem. Ann. Math. Statist. 27 (1956), 352–374.

Let $X_1, \cdots, X_m, X_{m+1}, \cdots, X_N$ be $N=m+n$ independent random variables, where X_i has the probability density $f_1(x)$ for $i=1, \cdots, m$ and $f_2(x)$ for $i=m+1, \cdots, N$. Let

$$t_N=(mn/N)^{\frac{1}{2}}(m^{-1}\sum_{i=1}^{m} b_{NR_i}-n^{-1}\sum_{i=m+1}^{m+n} b_{NR_i}),$$

where R_i denotes the rank of X_i in the set $X_1, \cdots, X_N$, and $b_{N1}, b_{N2}, \cdots, b_{NN}$ are real numbers. The paper is concerned with the asymptotic power of a test with the critical region $t_N>C$ (called an L test), of fixed size α for testing the hypothesis $f_1=f_2$, against the alternatives $f_i(x)=f_1(x, \delta N^{-\frac{1}{2}})$, $i=1, 2$, where $f_1(x, \theta)$ and $f_2(x, \theta)$ are given functions such that $f_1(x, 0)=f_2(x, 0)$, $\delta>0$, and $m/N\to K$, $(0<K<1)$, as $N\to\infty$. Part I: The most powerful rank test against θ sufficiently small (called the locally best rank order test — LBROT) is, under certain regularity conditions, an L test. Under further assumptions (including the asymptotic normality of t_N under the alternative), a theorem of Pitman yields the large sample power (=limit of the power for $N\to\infty$) of an L test. A partly heuristic argument suggests that the large sample power of the LBROT is equal to that of the most powerful test for testing $\theta=0$ against $\theta=\delta N^{-\frac{1}{2}}$, applied to suitably transformed variables $T(X_1), \cdots, T(X_N)$. Part II: An L test is called an L_h test if $b_{Ni}=p(i/N)$, where p is a polynomial of degree h. In this case t_N is shown to be asymptotically normal under general conditions. The large sample power of an L_h test is obtained, and the L_h test which maximizes the large sample power is determined. As $h\to\infty$, this maximum power tends, under mild restrictions, to the conjectured large sample power of the LBROT. *W. Hoeffding* (Chapel Hill, N.C.).

Rosenblatt, Murray. Remarks on some nonparametric estimates of a density function. Ann. Math. Statist. 27 (1956), 832–837.

Let $f(y)$ be a continuous probability density formation and let F_n be the empirical distribution function of a sample, size n, from f. For estimating $f(y)$ by the sequence of estimates $[F_n(y+h_n)-F_n(y-h_n)]/2h_n$ the asymptotic mean square error is evaluated as $n\to\infty$, $h_n\to 0$. With $h_n=kn^{-\alpha}$ it is shown to be a minimum when $\alpha=1/5$ and then to be of order $n^{-4/5}$. There is also a brief discussion of the estimates $f_n(y)=\int w_n(y-u)dF_n(u)$, where $\{w_n\}$ is a suitable sequence of probability density functions. *E. L. Lehmann* (Berkeley, Calif.).

Hájek, Jaroslav. Some rank distributions and their applications. Časopis Pěst. Mat. 80 (1955), 17–31. (Czech. Russian and English summaries)

The random variables α, β, γ are defined as follows: Let A_i $(i=1, 2, \cdots, 2^N)$ be all different subsets of $\{1, 2, \cdots, N\}$, each having the same probability 2^{-N}; denoting by A one of these subsets obtained at random, define α as the sum of the numbers in A. Similarly, let P_j $(j=1, 2, \cdots, N!)$ be all permutations of $[1, 2, \cdots, N\}$, each having the same probability $(N!)^{-1}$; denoting by P one of them obtained at random, define γ as the number of inversions in P. Let C_k $(k=1, 2, \cdots, \binom{N}{n})$ be all

combinations of n elements from $\{1, 2, \cdots, N\}$, each having probability $\binom{N}{n}^{-1}$; denoting by C one of these combinations obtained at random, define β as the number of inversions between C and its complement in $\{1, 2, \cdots, N\}$. — The probability distributions of these random variables are studied, also asymptotically for N large, and used for constructing tests of various hypotheses, some known (e.g. the Wilcoxon-Mann-Whitney test) and some new. *Z. W. Birnbaum* (Seattle, Wash.).

Raff, Morton S. On approximating the point binomial. J. Amer. Statist. Assoc. **51** (1956), 293–303.

A numerical comparison of six approximations to the binomial distribution. Criterion for goodness is the maximum error in all interval probabilities. It turns out that the Poisson-Gram-Charlier approximation is best when $np<0.8$, and the Camp-Paulson approximation [Camp, Ann. Math. Statist. **22** (1951), 130–131] when $np>0.8$. *G. Elfving* (Helsingfors).

Sugiyama, Hiroshi. Some theory of two-stage sampling inspection. Math. Japon. **3** (1955), 142–151.

Let an inspection lot be divided into M sublots of N items each. From these sublots m are drawn at random, and from each sampled sublot n items are drawn at random. Let $C_1, \cdots, C_M$ denote the numbers of defective items in the sublots prior to sampling, let $\dot{C}_1, \cdots, \dot{C}_m$ denote the numbers of defective items in the sampled sublots prior to sub-sampling, and let $\overline{C}$ and $\bar{C}$ denote the corresponding population and sample sublot means. Assuming $\bar{C}$ to be approximately normally distributed with mean $\overline{C}$ and variance σ_0^2 the author derives the probability of obtaining among the sampled items together at most c defective items. A somewhat different approach has been used by J. W. Hopkins [Bull. Inst. Internat. Statist. **34** (1955), 4ème livraison, 298–306]. *D. M. Sandelius* (Göteborg).

Laha, R. G. On the stochastic independence of two second-degree polynomial statistics in normally distributed variates. Ann. Math. Statist. **27** (1956), 790–796.

The main result is the following: Let $x=(x_1, x_2, \cdots, x_n)$ be n-variate normal with mean vector zero and variance-covariance matrix Σ. The necessary and sufficient condition that $P_1=xAx'+lx'$ and $P_2=xBx'+mx'$ are stochastically independent is that $A\Sigma B=0$, $l\Sigma B=0$, $m\Sigma A=0$, $l\Sigma m'=0$, where l and m respectively represent the row-vectors $(l_1, l_2, \cdots, l_n)$, $(m_1, m_2, \cdots, m_n)$, the primes denote transposes, and A and B are real symmetric matrices of order n. As a corollary it is shown that if the x's have different means but $\Sigma=I$, where I is the identity matrix, then the necessary sufficient condition for the stochastic independence of P_1 and P_2 is that

$$AB=0,\ lB=0,\ mA=0,\ lm'=0.$$

A second corollary result is that if the x's each have zero mean and $\Sigma=I$, then if P_1 and P_2 are stochastically independent there always exists an orthogonal transformation simultaneously reducing P_1 and P_2 to second degree polynomial statistics with no common variates. *S. Kullback* (Washington, D.C.).

Roy, S. N.; and Kastenbaum, Marvin A. On the hypothesis of no "interaction" in a multi-way contingency table. Ann. Math. Statist. **27** (1956), 749–757.

The authors consider a situation in which the independent observations are frequencies in a multi-way contingency table, subject to fixed total number only, from sample to sample. A hypothesis on the structure of the probabilities in the different cells or categories is put forward which by a certain analogy with the terminology of the analysis of variance is defined to be the hypothesis of "no interaction". For the three way table, which is discussed in detail, the hypothesis in question is

$$H_0: p_{ijk}=q_{ij0}q_{i0k}q_{0jk}/q_{i00}q_{0j0}q_{00k}$$
$$(i=1, 2, \cdots, r, j=1, 2, \cdots, s, k=1, 2, \cdots, t),$$

where the q's are essentially functions of the subscripts leading to $(r-1)(s-1)(t-1)$ constraints on the p_{ijk}'s. A large sample test of this hypothesis in terms of chi-square is derived. Bartlett's results for the case of a $2\times2\times2$ table [Suppl. J. Roy. Statist. Soc. **2** (1935), 248–252] and Norton's results for the case of a $2\times2\times t$ table [J. Amer. Statist. Assoc. **40** (1945), 251–258] formally turn out to be special cases of the results in this paper. *S. Kullback* (Washington, D.C.).

Takahashi, Masayuki. On analysis of variance for the split-plot designs. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 321–325.

The purpose of this paper is to make clear the mechanism of analysis of variance for the split-plot designs by rigorous treatments applicable to general cases. The standpoint of the author is to appeal to the notions of orthogonal classifications, projective operators, which serve to make clear at least algebraic features of the analysis of variance. {It is the opinion of the reviewer that split-plot designs characterised by the condition (1) in this paper seem more general than those defined in usual sense.} *T. Kitagawa* (Fukuoka).

Graybill, Franklin A.; Martin, Frank; and Godfrey, George. Confidence intervals for variance ratios specifying genetic heritability. Biometrics **12** (1956), 99–109.

In a breeding experiment, r sires are each mated at random to s dams, and t offspring result from each mating. The linear model $Y_{ijk}=\mu+a_i+b_{ij}+c_{ijk}$ is considered, where $i=1, 2, \cdots, r$; $j=1, 2, \cdots, s$; and $k=1, 2, \cdots, t$; a_i, b_{ij}, c_{ijk} are distributed respectively as $N(0, \sigma_a^2)$, $N(0, \sigma_b^2)$ and $N(0, \sigma_e^2)$. A linear function of the mean squares between sires and between dams (within sires) is obtained, having the same first two moments as a χ_N^2 distribution, where N is a function of the unknown parameters for which, in genetical examples, plausible bounds are available. Division of the χ_N^2 variate by the residual mean square gives a multiple of an F-variate, and confidence limits are obtained for

$$2(\sigma_a^2+\sigma_b^2)/(\sigma_a^2+\sigma_b^2+\sigma_e^2).$$

Extensive sampling experiments suggest that the actual confidence coefficient is fairly close to the nominal value. *P. Armitage* (London).

Bunkin, F. V. On the question of the statistical character of transmagnetisation of ferromagnetic materials. 2. Tehn. Fiz. **26** (1956), 1782–1789. (Russian)

Rybarz, J. Neue Wege der modernen Risikotheorie. Österreich. Ing.-Arch. **10** (1956), 260–264.

This is an exposition — without proof — of B. de Finetti's theorem [Giorn. Ist. Ital. Attuari **10** (1939), 41–51] on the gamblers ruin. *E. Lukacs* (Washington, D.C.).

See also: Ryser, p. 105; Hussain, p. 105; Kahn, p. 151; Trotter, p. 152; Arnold, Bucher, Trotter and Tukey, p. 153; Marshall, p. 153; Walsh, p. 153; Lewontin and Prout, p. 180.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Falk, S. Die Berechnung des beliebig gestützten Durchlaufträgers nach dem Reduktionsverfahren. Ing.-Arch. **24** (1956), 216–232.

Für die Berechnung von statisch unbestimmte Tragwerke kann man entweder die klassischen Eliminationsmethoden (Kraftmethode und Deformationsmethode) oder die in den letzten Jahren öfters benutzte Reduktionsmethode anwenden. An drei Beispielen wird die letzte vom Verf. vorgeführt und der Rechenaufwand mit dem der Eliminationsmethode verglichen. Es wird dabei eine fünfreihige Matrix benutzt, welche die Werte der bezüglichen Grössen vom linken zum rechten Feldrande hinüber leitet; die Rechnung ist so weit schematisiert dasz sie auf Rechenautomaten durchgeführt werden kann.

O. Bottema (Delft).

Sieker, K.-H. Zur algebraischen Masssynthese ebener Kurbelgetriebe. I. Ing.-Arch. **24** (1956), 188–215.

Soll von einer Welle eine Drehbewegung auf eine parallele Welle in der Weise übertragen werden daß der zweite Drehwinkel eine gegebene Funktion des ersten ist, so kann nur ein Kurvengetriebe die genaue Lösung geben. Will man ein Gelenkviereck verwenden so können nur fünf Punkte der gegebenen Funktion erfüllt werden. Es gibt geometrische Verfahren, welche auf Burmester zurückgehen um das bezügliche Viereck zu konstruieren. Die fünf Punkte können zusammenrücken so daß der Funktionswert und die Ableitungen bis zur vierten Ordnung gegeben sind. Auch eine Kombination von Punkten mit endlichem und infinitesimalen Abstand ist möglich. Für jeden Gelenkpunkt findet man die Burmesterschen Punkte (ein oder drei). In dieser Arbeit werden nun diese Zusammenhänge algebraisch mittels komplexer Zahlen dargestellt; für vier Punkte hat der Verf. das schon früher getan [Verein. Deutsch. Ing. Ber. **5** (1955)]. Die Rechnungen werden durch ausführliche Zahlenbeispiele erläutert.

O. Bottema (Delft).

Lichtenheldt, Willibald. Die Methode der Partialsynthese. Wiss. Z. Tech. Hochsch. Dresden **5** (1955/56), 79–82.

Es wird an einigen Beispielen gezeigt wie durch schrittweise Anwendung von einfachen geometrischen Methoden eine Getriebeaufgabe gelöst werden kann. Eins dieser Beispiele befasst sich mit der Konstruktion eines Filmgreifergetriebes für ein vorgegebenes Zeitverhältnis.

O. Bottema (Delft).

★**Binder, R. C. Mechanics of the roller chain drive.** Prentice-Hall, Inc., Englewood Cliffs, N.J., 1956. Distributed by Diamond Chain Company, Inc., 402 Kentucky Avenue, Indianapolis 7, Indiana. xiii+204 pp. $5.00.

Rašković, Danilo P. On some characteristics of the frequency equation of small vibrations of holonomic conservative systems with static couplings. Quart. Appl. Math. **14** (1956), 309–311.

The author obtains explicitly the frequency equations and the frequencies for three particular types of systems of n coupled equal masses oscillating linearly.

P. Franklin (Cambridge, Mass.).

Požarickiĭ, G. K. On non-steady motion of conservative holonomic systems. Prikl. Mat. Meh. **20** (1956), 429–433. (Russian)

Let S be a solution of the Hamiltonian system $\dot{p}_i=-\partial H/\partial q_i$, $\dot{q}_i=\partial H/\partial p_i$, $1\leqq i\leqq n$, where H is holomorphic in p_i, q_i and independent of t, and consider along S the $k\times k$ matrices $P_k=(\partial^2 H/\partial p_\alpha \partial p_\beta)$, $Q_k=(\partial^2 H/\partial q_\alpha \partial q_\beta)$, where $1\leqq\alpha,\beta\leqq k$ and $1\leqq k\leqq n$. By Lyapunov's first theorem on instability [cf., e.g., Problème général de la stabilité du mouvement, Princeton, 1947, p. 262; MR **9**, 34], S is unstable if $P_n>0$ and $Q_n<0$. The author shows that, when $Q_n\equiv \operatorname{diag}(Q_j, Q_{n-j})$ for some j, $1\leqq j\leqq n$, S is unstable if $P_j>0$ and $Q_j<0$.

H. A. Antosiewicz.

See also: Četaev, p. 128; Krasovskiĭ, p. 128; Green, p. 162.

Statistical Mechanics

Meijer, Paul H. E. Extension of the statistical proof of the minimum entropy production theorem. Phys. Rev. (2) **103** (1956), 839–844.

Klein and Meijer gave a proof of the minimum entropy production in the steady state based on the principles of statistical mechanics, making use of a simple model of a nonequilibrium system. In the first part of this paper it is shown how such a proof has to be modified in case we are dealing with particles that obey Fermi-Dirac or Bose-Einstein statistics. It is found that the constraint that the total number of particles has to be constant does not have to be made. The second part of the paper describes a different generalization, viz. to systems in which the total number of particles is variable. Both results will serve as a description for models less restrictive than that used in the paper mentioned above. *Author's summary.*

Pignedoli, Antonio. Su alcuni problemi diffusivi della fisica matematica e su una questione connessa di meccanica statistica, interessante la teoria del moto browniano. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) **1** (1953–54), no. 2, 5–20.

The following equations occurring in the theory of neutron diffusion, and the methods to solve them, are reviewed. (i) Diffusion of a thermal neutron gas in a finite medium (method of stationary solutions). (ii) Multiple scattering in a one-dimensional medium (method of Laplace transforms). (iii) Slowing down of neutrons (Fokker-Planck equation). *N. G. van Kampen* (Utrecht).

Sarolea, Liliane; and Mayer, Joseph E. Integral equation method in the theory of liquids. Phys. Rev. (2) **101** (1956), 1627–1640.

In the theory of a liquid of several components, Mayer's integral equations [J. Chem. Phys. **15** (1947), 187–201] establish a rigorous relation between the hierarchy of molecular distribution functions at two different densities. They are here cast in a form from

which various known results, in particular the equations of Kirkwood and of Yvon-Born-Green, obtain by suitable specialization. If the Kirkwood approximation is used, a closed set of equations results, but if one tries to improve upon this approximation by adding successive terms, the resulting series cannot be expected to converge above the condensation density. Other methods of approximately solving the rigorous equations are suggested and discussed. *N. G. van Kampen* (Utrecht).

See also: Bopp, p. 173; Gombás, p. 175.

Elasticity, Visco-elasticity, Plasticity

Truesdell, C. Das ungelöste Hauptproblem der endlichen Elastizitätstheorie. Z. Angew. Math. Mech. **36** (1956), 97–103. (English, French and Russian summaries)

This is the printed version of a lecture in which the author called attention to the following problem: In the classical theory of finite elastic deformations the mechanical properties of a body are described by giving the strain energy Σ as a function of Green's deformation tensor. Most of the remarkable results obtained in this theory in recent years are valid for an arbitrary choice of the functional form of Σ. However, it is obvious that Σ must be subject to restrictions in the form of inequalities in order that physically unacceptable behavior, such as production instead of consumption of energy in loading processes, is excluded. The main open problem consists in finding the precise class of admissible functions Σ.

The author reviews the history of this problem, which starts with Hadamard's research on the existence of acceleration waves, and he discusses the relation of the problem to questions of uniqueness and stability of solutions of boundary value problems.

The precise restriction on Σ will probably be derived eventually from the entropy principle, a version of which general enough to be applicable to general continuum mechanics remains to be discovered, too. *W. Noll.*

★ **Huber, M. T. Teoria sprężystości. Część I. [Theory of elasticity. Part I.] 2d ed.** Państwowe Wydawnictwo Naukowe, Warszawa, 1954. xiii+375 pp. zł. 43.35.

The present second edition of the Theory of Elasticity of the late Professor M. T. Huber appears as volumes IV and V of his Collected Works, the publication of which has been sponsored by the Polish Academy of Sciences. This edition, edited by a special committee of the Academy, is a carefully revised and corrected version of the first edition written during the war years that was published in 1948/49, during the author's life-time.

The work is divided into two volumes. The first three chapters of the first volume deal with the fundamental aspects of the states of stress and strain and their interrelation, as the basis of the theory of elasticity. Chapters IV and V present an extensive analysis of two-dimensional problems in rectangular and polar coordinates, while Chapter VI treats the solution of plane problems by the aid of complex variables. Chapter VII is devoted to a general discussion of energy methods, Chapter VIII deals with simple problems of the distribution of concentrated forces within the elastic medium, while Chapter IX considers some elementary three-dimensional problems. Chapters X and XI are devoted to a detailed discussion of problems of torsion, including the use of mathematical analysis and energy methods in their solution.

The treatment is complete and meticulous, with a wealth of particular problems treated in detail. It is a text-book written for engineering and applied mechanics students and research workers rather than for applied mathematicians. *A. M. Freudenthal.*

★ **Huber, M. T. Teoria sprężystości. Część II. [Theory of elasticity. Part II.] 2d ed.** Państwowe Wydawnictwo Naukowe, Warszawa, 1954. ix+357 pp. zł. 43.20.

The first chapter of the second volume (Chapter XII) presents a short introductory treatment of vibration and wave-propagation problems, chapters XIII and XIV deal with bodies of rotational symetry and the associated stress functions, including an extensive discussion of the functions of Neuber for the solution of the state of stress around notches of rotational symmetry. Chapter XV is concerned with the elastica and with the vibration of and wave-propagation in strings and beams. Chapter XVI deals extensively with the theory of isotropic plates and the various methods for the solution of particular problems, while Chapter XVII presents a review of the author's original contributions to the theory of the anisotropic (orthotropic) plate. Chapter XVIII deals extensively with the membrane theory of thin shells of general form, and discusses the bending theory of cylindrical shells and of thick-walled shells of simple shape, as well the general equations of the theory of shells of arbitrary shape. The final Chapter XIX reviews the physical basis of the theory of plasticity, illustrating its application by the problem of "auto-frettage".

As in the first volume the treatment is on the level of engineering mechanics rather than of applied mathematics. Within this limitation of purpose, however, it is an excellent text-book, probably the most complete text of its kind this side of Love, born of the author's long experience as a teacher and reflecting his status as one of the most prominent "applied mechanicians" of his time. *A. M. Freudenthal* (New York, N.Y.).

Green, A. E. Simple extension of a hypo-elastic body of grade zero. J. Rational Mech. Anal. **5** (1956), 637–642.

The author obtains special exact solutions of the equations of Truesdell's theory of hypo-elastic materials of grade zero [same J. **4** (1955), 83–133; MR **16**, 880]. These solutions are valid for compressible bodies and they correspond to a special type of simple extension. Comparing his results with those obtained after neglecting inertia terms he observes that the effect of inertia can legitimately be neglected. However, this is not the case for other types of solutions [cf. the paper cited above]. *W. Noll* (Pittsburgh, Pa.).

Budiansky, Bernard; and Pearson, Carl E. On variational principles and Galerkin's procedure for non-linear elasticity. Quart. Appl. Math. **14** (1956), 328–331.

The third boundary value problem of elasticity is considered in the case where the strains are taken linear in the displacement gradients, but the stress-strain relations are derived from an arbitrary strain energy function. H. J. Greenberg [Grad. Div. Appl. Math., Brown Univ., Tech. Rep. **All-S4** (1949); MR **11**, 559] mentions two variational principles that are satisfied by any solution of the problem. The first of these is the minimum energy principle satisfied by the displacements.

The present paper gives an easy proof of the converse statement, namely that a displacement vector satisfying this principle is a solution. This is the derivation of equilibrium and boundary conditions by means of virtual work.

The other principle mentioned by Greenberg is a generalization due to F. Engesser [Z. Architekten- und Ingenieurvereins Hannover **35** (1889), 734–742] of Castigliano's principle. The converse statement, that a system of stresses satisfying this variational principle leads to compatible strains satisfying the boundary conditions is easily proved by means of Lagrange multipliers [H. L. Langhaar, J. Franklin Inst. **256** (1953), 255–264, p. 263; MR **15**, 266]. Apparently unaware of this, the authors give a rather difficult proof of the latter converse statement. The same basic lemma as that used by the authors has also been proved by W. S. Dorn and A. Schild and published, curiously enough, in the preceding issue of the same journal [Quart. Appl. Math. **14** (1956), 209–213; MR **18**, 83]. *H. Weinberger.*

Eubanks, R. A.; and Sternberg, E. On the completeness of the Boussinesq-Papkovich stress functions. J. Rational Mech. Anal. **5** (1956), 735–746.

The authors investigate the representation of the displacement field $\mathbf{u}$ of a linearly elastic isotropic body without body forces in terms of the Boussinesq-Papkovich harmonic displacement potentials φ and ψ in the form

$$\mathbf{u}=\operatorname{grad}(\varphi+\mathbf{r}\cdot\psi)+4(1-\nu)\psi, \tag{1}$$

where $\mathbf{r}$ is the position vector and ν is Poisson's ratio. First they state precise conditions on $\mathbf{u}$ and its domain of definition $\mathcal{D}$ under which such a representation is always possible. The main part of the paper deals with the possibility of taking φ or components of ψ as zero in (1). Results: 1) If $\mathcal{D}$ is star shaped and if 4ν is not an integer, φ may be omitted in (1). A counterexample is given to show that this is in general not the case if 4ν is an integer. 2) If $\mathcal{D}$ satisfies certain conditions it is possible to take one component of ψ as zero in (1). 3) If $\mathbf{u}$ is axisymmetric and if $\mathcal{D}$ is a solid of revolution then it is possible to choose both φ and ψ as axisymmetric and ψ as having only a component in the direction of the axis. *W. Noll.*

Hristoforov, V. V. Application of Green's functions to plane problems of the static theory of elasticity. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **15** (1955), 131–141. (Russian)

For the first boundary value problem the displacements at interior points are expressed in terms of their position and of the dilatation Θ on the boundary by means of Green's function for the Dirichlet problem. An integral equation for the boundary values of Θ is then obtained. A similar equation is given for the rotation ω. For the second boundary value problem, Green's function for the Neumann problem is used and a pair of integral equations for ω and Θ is obtained. *R. C. T. Smith* (Armidale).

Moskvitin, V. V. Elastic-plastic torsion of shafts under repeated loading. Vestnik Moskov. Univ. **11** (1956), no. 3, 31–40. (Russian)

A procedure is outlined for computing the torque-twist relations of uniform shafts containing residual stresses. Reversed or repeated loading programs can be followed. Increments of elastic deformation are assumed to be the same as those in a shaft which has not been deformed plastically. As an example, the procedure is applied to the shaft of oval cross-section in which an elliptical elastic-plastic interface develops. The solution in the case of zero initial stress has been given by V. V. Sokolovsky [Prikl. Mat. Meh. **6** (1942), 241–246; MR **4**, 180]. Sokolovsky's solution is not valid for an intermediate range of twist-rates during which the elastic-plastic interface intersects the external contour. The author completes his torque-twist cycles in this range by sketching curves which join smoothly with the neighboring computed parts.

R. M. Haythornthwaite (Providence, R.I.).

Ghosh, S. Torsion of a solid of revolution of a material possessing curvilinear aeolotropy. J. Assoc. Appl. Phys. Calcutta Univ. **3** (1956), 1–4.

Šerman, D. I. On bending of a circular plate, partly supported and partly free on the contour. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1180–1183. (Russian)

A plate simply supported over half its boundary, free over the remainder, and subjected to uniform normal pressure is considered. The deflection is expressed in terms of two complex functions ϕ, ψ and the boundary conditions reduced to functional equations in ϕ, ψ. It is easy to calculate ϕ if ψ is known. The equation for ψ is equivalent to an integral equation [see Mushelišvili, Singular integral equations, Gostehizdat, Moscow-Leningrad, 1946; MR **8**, 586; **15**, 434]. Substituting a power series for ψ leads to a system of equations in infinitely many unknowns for the coetficients and this system may be shown to be completely regular. *R. C. T. Smith.*

Chakravorty, J. G. On the twisting of a spherical shell of a spherically aeolotropic material (in the broader sense). J. Assoc. Appl. Phys. Calcutta Univ. **3** (1956), 14–16.

Kopzon, G. I. The vibration of thin-walled elastic bodies in a gas stream. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 217–220. (Russian)

Consider a long slender shell of revolution (or fuselage) internally braced by periodically spaced stringers and immersed in a compressible flow at Mach number M_1. Suppose that under the aerodynamic load due to its own vibrations the shell performs harmonic oscillations such that in cylindrical coordinates y, r, ϕ the radial displacements $w(y, \phi, t)$ are of the form

$$w=\sum_0^\infty q_n(y)\exp i(\omega_n t+n\phi). \tag{1}$$

By means of linearized compressible flow theory the excess pressure for the surface defined by (1) can be determined approximately and inserted in the partial differential equation of a vibrating shell which is in effect a boundary condition for the compressible flow problem. For solutions of the form (1) and for supersonic M_1 this yields a system of integro-differential equations of convolution type for $q_n(y)$ which can be solved by means of Laplace transforms, while the frequencies ω_n are found from the boundary conditions at the ends of the shell. The author calculates the drag coefficient for the drag due to the vibrations. He also discusses extensions of this treatment to more general shells and sketches a proof that the solution of the general motion governed by integro-differential equations of convolution type is unique. *J. Giese* (Aberdeen, Md.).

Bland, D. R. Elastoplastic thick-walled tubes of work-hardening material subject to internal and external pressures and to temperature gradients. J. Mech. Phys. Solids **4** (1956), 209–229.

"Using Tresca's yield criterion and its associated flow rule, solutions are obtained for the stresses, the elastic and plastic strains and the displacements when a thick-walled tube of work-hardening material is subject to internal and external pressures and its surfaces are maintained at different temperatures. In general a numerical integration is necessary, but the solutions can be expressed explicitly when the work-hardening law is linear. On removal of the pressures and the temperature difference plastic flow may or may not occur. In both cases the residual stresses can be calculated. Specific examples are given." (Author's summary.)

The variations in temperature are assumed to be small enough that the yield stress and all other relevant physical properties may be considered as constant. Furthermore, the tube is presumed long enough that stress and strain do not vary with axial position.

D. C. Drucker (Providence, R.I.).

Mechovrišvili, Š. S. Some questions of the momentless stressed state of a toroidal shell. Soobšc. Akad. Nauk Gruzin. SSR **16** (1955), 263–267. (Russian)

Membrane theory for a toroidal shell leads to

$$\frac{\partial}{\partial\phi}\left[(1-\varepsilon\sin\phi)\frac{\partial W}{\partial\phi}\right]-\frac{\varepsilon}{\sin\phi}\frac{\partial^2 W}{\partial\theta^2}+(1-2\varepsilon\sin\phi)W=0,$$

an equation of some interest as it is hyperbolic for $0<\phi<\pi$, elliptic for $\pi<\phi<2\pi$. Analytic solutions satisfying appropriate boundary conditions are discussed and momentless stress systems of physical interest are shown not to exist. *R. C. T. Smith* (Armidale).

Buzin, E. I. On the stresses in a surface described by a moving circle perpendicular to its curve of centers and in equilibrium under the action of constant internal pressure forces. Dokl. Akad. Nauk Uzbek. SSR. **1954**, no. 1, 3–8. (Russian. Uzbek summary)

The author derives the equation of the surface indicated in the title and using known equations of equilibrium of a flexible unstretchable surface obtains formulas giving the stresses. In particular, stresses in a surface of a torus are obtained. *A. P. Hvoles* (RŽMat 1955, no. 5218).

Olszak, W.; et Życzkowski, M. Les bases de la théorie de l'élasticité des corps physicalement non-linéaires à structure non-homogène. Arch. Mech. Stos. **7** (1955), 151–168. (Polish. Russian and French summaries)

The stress-strain relations are established and discussed of the non-linear elastic continuum of both initial and strain-dependent inhomogeneity considering small strains, under the simplifying assumption of constant Poisson ratio. The resulting differential equations, containing the gradients of the space – and strain – dependent modulus, are non-linear and with variable coefficients.

The problems are discussed of torsion of a circular bar with radial inhomogeneity and of the longitudinal vibration of a thin bar with axial inhomogeneity.

A. M. Freudenthal (New York, N.Y.).

Laval, Jean. L'énergie potentielle et l'élasticité du milieu cristallin. C. R. Acad. Sci. Paris **242** (1956), 2502–2505.

The classical theory of elasticity gives the potential energy of a crystal as a sum of bi-atomic terms. The assumptions involved are that the atoms are spherical, the electron density is uniform and the forces are central. Without these assumptions it is found that the potential energy may contain tri-atomic and higher terms. A number of results that follow are given without proof.

B. R. Seth (West Bengal).

Adkins, J. E. Finite plane deformation of thin elastic sheets reinforced with inextensible cords. Philos. Trans. Roy. Soc. London. Ser. A. **249** (1956), 125–150.

This paper constitutes another considerable advance of the theory of finite elastic deformations. As a preliminary, the author derives the general equations for the deformation of curvilinearly aeolotropic elastic materials subject to arbitrary constraints, thus generalizing the work of Ericksen and Rivlin [J. Rational Mech. Anal. **3** (1954), 281–301; MR **16**, 88], who considered the case of rectilinear aeolotropy only.

He then considers thin plane sheets reinforced by two layers of thin, flexible, inextensible cords located in the middle plane and following the paths of two arbitrary families of curves. The deformations under investigation can be described as plane deformations of the middle plane superposed to extensions perpendicular to the middle plane. The elastic properties of the sheet are assumed to be symmetric with respect to the middle plane. Under these circumstances the presence of the layers of cords is equivalent to two constraints prevailing in the entire sheet, and hence the theory developed in the beginning can be applied. [This method of dealing with reinforced sheets is somewhat different from that employed by the author in his previous work on reinforced elastic bodies, cf. Adkins and Rivlin, Philos. Trans. Roy. Soc. London Ser. A. **248** (1955), 201–223; Adkins, J. National Mech. Anal. **5** (1956), 189–202; MR **17**, 427, 1025]. The general equations reduce to a system of two hyperbolic differential equations for the two displacement components of the middle plane, the characteristic curves being the paths of the cords. For any solution of this system, the stresses can be determined by solving another such hyperbolic equation for an appropriate Airy's stress function. For the case that the cords lie along two sets of parallel straight lines before deformation the author derives formulae expressing the displacements, the stresses, and the force acting across a curved cut of the sheet in terms of four arbitrary functions. As an application, he considers the deformation of an infinite plane sector produced by applying forces along one edge and prescribing the displacement along the other.

The remainder of the paper contains a brief discussion of sheets in which only one layer of cords is present and of the simplifications arising in the case of small deformations. *W. Noll* (Pittsburgh, Pa.).

Melvin, M. A.; and Edwards, S., Jr. Group theory of vibrations of symmetric molecules, membranes, and plates. J. Acoust. Soc. Amer. **28** (1956), 201–216.

The paper reformulates the familiar group-theoretical method of classifying molecular vibrations so that it can be applied to continuous bodies, membranes or plates, with given symmetries. The case of continuous bodies is treated as limit of discrete systems composed of a large number of particles. One must in addition include possible constraints, which are usually present for membranes or plates but never for molecules. *L. Van Hove* (Utrecht).

Kopzon, G. I. Harmonic oscillations of elastic plates in a subsonic flow. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 2, 83–92. (Russian)

In a subsonic plane flow parallel to the x-axis consider a vibrating wing whose structure has period l in the spanwise (z-) direction. The author replaces this by a flat plate of the same chord $2b$ whose displacements satisfy $[(\partial^2/\partial x^2+\partial^2/\partial y^2)^2+(\gamma h/Dg)\partial^2/\partial t^2]w=\Delta p/D$ and seeks the characteristic frequencies. General series forms, with undetermined coefficients, for the local excess pressure $\Delta p(x, z, t)$ on a surface harmonically oscillating with circular frequency ω are given in L. I. Sedov's book "Plane problems of hydrodynamics and aerodynamics" [Gostehizdat, Moscow, 1950] and E. A. Krasil'ščikova's "Wing of finite span in a compressible flow" [Gostehizdat, Moscow-Leningrad, 1952; MR 16, 537]. Let

$$w(x, z, t)=e^{i\omega t}\sum_{-\infty}^{\infty} e^{2n\pi iz/l}w_n(x)$$

and

$$\Delta p=e^{i\omega t}\sum_{-\infty}^{\infty} e^{2n\pi iz/l}\Delta p(w_n).$$

Then

$$(*)\qquad [(d^2/dx^2-4\pi^2n^2/l^2)-m_b]w_n(x)=m\Delta p(w_n),$$

where m_b is a constant multiple of $b^2\omega^2$ and m is a small constant that depends on fluid density, velocity, plate dimensions, and elastic constants. Now let $w_n(x)=\sum_{s=0}^{\infty}(m)^s w_s^{(n)}(x)$ and $\Delta p(w)=\sum_0^{\infty}(m)^s\Delta p(w_s^{(n)}(x))$. The general solution $w_0^{(n)}(x)$ of the homogeneous form of (*) corresponds to a plate vibrating in a vacuum and is easily found. Once $w_s^{(n)}(x)$ is known, $\Delta p(w_s^{(n)}(x))$ can be found, and then $w_{s+1}^{(n)}(x)$ from (*). As usual, the boundary conditions for the appropriate types of support at the leading and trailing edges $x=\pm b$ determine the characteristic values of ω. Now write $\omega=\sum_0(m)^s\omega_s$, where ω_0 is a characteristic value for vibration in a vacuum. For small m, to avoid extremely unwieldy manipulations, the author proposes to use the first approximations $w_0^{(n)}(x)+mw_1^{(n)}(x)$, $\omega_0+m\omega_1$, and describes the detailed computation required to find them. *J. Giese.*

Lenskiĭ, V. S. On Ostrogradskiĭ's method of integrating the dynamical equations of the theory of elasticity. Prikl. Mat. Meh. **19** (1955), 617–620. (Russian)

The equations of motion for an isotropic solid are solved giving closed expressions for the displacements similar to Poisson's solution of the wave equation. The solid is supposed infinite, acted on by body forces varying both with time and position, and with given values of the initial displacements and velocities.

R. C. T. Smith (Armidale).

Olszak, Wacław. Sur les bases de la théorie des corps élasto-plastiques non-homogènes. II. Arch. Mech. Stos. **6** (1954), 639–656 (1955). (Polish. Russian and French summaries)

[For part I see same Arch. **6** (1954), 493–532; MR **16**, 542.] The stress-strain relations limited to small strains are presented for the inhomogeneous elastic-plastic body with initial space dependence of both the elastic and the plastic parameters, including the yield condition. A classification of inhomogeneous elastic-plastic response is attempted. *A. M. Freudenthal* (New York, N.Y.).

See also: Womersley, p. 166; Narasimhan, p. 166; Ross, p. 166.

Fluid Mechanics, Acoustics

Kostyčev, G. I. On potential flow about a profile near a plane boundary. Kazan. Aviac. Inst. Trudy **29** (1955), 25–37. (Russian)

Let L be a thick airfoil in the upper half of the $z=x+iy$ plane. Let L' be its image with respect to the x-axis. Now some function

$$z(\zeta)=A\xi+\sum_1^{\infty}C_n(\zeta-IR)^{-n}+\sum_1^{\infty}\bar{C}_n(\zeta+iR)^{-n},$$

where A is a real constant and C_n and $\bar{C}_n$ are conjugate complex constants, maps the exterior of L and L' onto the exterior of the unit circles $\zeta=\pm iR+e^{i\theta}$ in the $\zeta=\xi+i\eta$ plane, for $R>1$. Suppose the function $z_1(\zeta_1)=c\zeta_1+\sum_0^{\infty}c_n\zeta_1^{-n}$ which maps the exterior of L onto the exterior of $\zeta_1=e^{i\gamma}$ is known. To determine $z(\zeta)$ approximately the author assumes that A and C_n are analytic functions of $\lambda=1/2R$, and that $\gamma=\theta+\sum_3^{\infty}\lambda^n f_n(\theta)$, where $z=z(iR+e^{i\theta})=z(e^{i\gamma})$ on L, and $f_n(\theta)$ have period 2π. He determines $f_3, \cdots, f_6$ and also $A, C_1, \cdots, C_9$ to terms of order λ^6 as functions of $c, c_1, \cdots, c_5$. With the corresponding approximation to $z(\zeta)$ and the known complex velocity potential for incompressible flow parallel to the ξ-axis about the two unit circles he can then construct an approximate complex potential for flow parallel to the x-axis about L and L'.

J. H. Giese.

Sretenskiĭ, L. N. On directed emission of waves from a region subjected to external pressure. Prikl. Mat. Meh. **20** (1956), 349–361. (Russian)

The half-space $z<0$ is occupied by a liquid, the surface $z=0$ of which is subject to a pressure $p=p_0\exp(i\omega t)$ ($|x|<a$, $|y|<b$), $p=0$ elsewhere on the surface. The boundary-value problem is thus the solution of Laplace's equation $\Delta\phi=0$ for the velocity potential ϕ, with $\phi_{tt}+g\phi_z=\varrho^{-1}p_t$ when $z=0$ and $\phi=0$ when $z=-\infty$. The ordinate $\zeta(x, y; t)$ of the resulting surface wave is given in terms of a Fourier integral, which is evaluated by contour integration. It is shown that undamped plane waves are propagated from the region under pressure in the bands $|x|<a$ and $|y|<b$, and that in the remaining parts of the surface the amplitude of the oscillations diminishes like $R^{-1}=(x^2+y^2)^{-\frac{1}{2}}$ for large R. Finally is it proved that to sustain the sequence of progressive waves it is sufficient to impart to the liquid an energy equal to half the energy of the waves. *R. N. Goss* (San Diego, Calif.).

Peržnyanko, È. A. On vertical oscillations of a body floating on the surface of a liquid between two parallel walls, and on waves produced in this situation. Prikl. Mat. Meh. **20** (1956), 362–372. (Russian)

An infinitely deep liquid is contained between two parallel vertical walls. Midway between them on the surface floats a body symmetric about the vertical axis through the centroid. A slight displacement of the body in the vertical direction results in oscillations which are communicated to the surrounding liquid. The author determines the equation of motion of the centroid and obtains a solution in the form of an infinite series of residues. He studies particularly the beating phenomenon which takes place when the natural frequency of oscillation of the body is near one of the standing-wave frequencies in the liquid due to the presence of the bounding walls.

R. N. Goss (San Diego, Calif.).

Bass, Jean. Sur les solutions des équations du mouvement d'un fluide visqueux incompressible en écoulement non permanent. C. R. Acad. Sci. Paris **243** (1956), 229–230.

Consider a viscous, incompressible, unsteady flow in a convex simply connected domain D, bounded by a fixed surface S. It is shown that if the velocity u_i and the pressure p satisfy certain initial conditions in D, and boundary conditions on S, and if $u_i(x_j, 0)$, and $u_i(x \epsilon S, t)$, $p(x \epsilon S, t)$ are sufficiently small a solution of the equations of motion exists.

The argument is based on a consideration of the associated system (to the Navier-Stokes equations) $\partial u_i/\partial t + \lambda(u_k \partial u_i/\partial x_k + \partial p/\partial x_i) = \mu \Delta u_i$ and an expansion of u_i and p in a power series in λ. The equation for u_i^0 is simply the heat equation, and under appropriate conditions the radius of the power series is greater than one.

R. C. DiPrima (Culver City, Calif.).

Stewartson, K. On the steady flow past a sphere at high Reynolds number using Oseen's approximation. Phil. Mag. (8) **1** (1956), 345–354.

The flow of a viscous fluid past a sphere at high Reynolds number as given by Oseen's approximation, in which the inertia terms of the equations of motion are linearized, is discussed. It is found that a boundary layer is formed on the forward side and a wake extending to infinity on the rearward side. Inside the wake, which is bounded by a circular cylinder with its generators parallel to the undisturbed direction of the streamlines, the fluid at infinity is moving towards the sphere, while outside it is moving away with the undisturbed velocity, and there is a shear layer at the cylinder. (Taken from the author's summary.) *R. C. DiPrima* (Culver City, Calif.).

Womersley, J. R. Oscillatory motion of a viscous liquid in a thin-walled elastic tube. I. The linear approximation for long waves. Phil. Mag. (7) **46** (1955), 199–221.

Die vorliegende Untersuchung wurde durchgeführt, um über die Bewegung des Blutes in den Arterien Aufschluss zu erhalten. Deshalb wurde die Strömungsgeschwindigkeit vorerst auf so kleine Werte begrenzt, dass die Trägheitsglieder der Navier-Stokes'schen Gleichung vernachlässigt werden konnten. Die durch die Berücksichtigung der Trägheitsglieder erforderliche Korrektur soll in einem zweiten Teil ermittelt werden. Die grundsätzlichen Zusammenhänge werden dadurch nicht verändert. Eine Vorbetrachtung der zähen Strömung in einem starren Rohr bei periodisch veränderlichem Druck ergab Grösse und Phasenlage der dadurch hervorgerufenen Flüssigkeitsbewegung. Der Übergang zum elastischen Rohr hatte nicht nur eine Dämpfung dieser Bewegung sondern auch eine Veränderung der Wellen-Fortpflanzungsgeschwindigkeit zur Folge. Die Dämpfung wird vergrössert durch eine Vergrösserung der Zähigkeit sowie eine Verringerung der Frequenz und der Poissonschen Zahl. Die Wellen-Fortpflanzungsgeschwindigkeit wächst mit zunehmender Frequenz. Bei konstanter Frequenz steigt die Fortpflanzungsgeschwindigkeit mit Abnahme der Zähigkeit und nähert sich asymptotisch dem von Lamb [Mem. Proc. Manchester Lit. Philos. Soc. **42** (1897–98), no. 9] ermittelten Wert. Weiter wird gezeigt, dass die Längsschwingung der Rohrwand, die durch die Wandreibung der Flüssigkeit hervorgerufen wird, die Durchflussmenge gegenüber dem starren Rohr bei gleichem Druckgradienten um etwa 10% vergrössert. *L. Speidel.*

Narasimhan, M. N. L. On the steady laminar flow of certain non-Newtonian liquids through an elastic tube. Proc. Indian Acad. Sci. Sect. A. **43** (1956), 237–246.

This paper is a study of the effect of cross-viscosity on the radius of an elastic tube when certain highly viscous non-Newtonian liquids flow through it under pressure gradient. The effect is found to be of opposite nature to that of inertial and viscous forces and causes the radius of the tube to increase with distance along the axis. (Author's summary.) *Y. H. Kuo* (Peking).

★ **Шлихтинг, Г. Теория пограничного слоя. [Šlihting, G. Boundary layer theory.]** Translated by G.A. Vol'pert. Izdat. Inostr. Lit., Moscow, 1956. 528 pp. 30 rubles.

A translation of Grenzschicht-Theorie [Braun, Karlsruhe, 1951; MR **13**, 177] with interspersed reference to Soviet literature and with a bibliography added at the end.

Ross, Donald. Turbulent flow in the entrance region of a pipe. Trans. A.S.M.E. **78** (1956), 915–921, discussion 921–923.

This paper deals with a method for the approximate solution of turbulent boundary-layer flow in the entrance region of a pipe. Assuming that the boundary-layer becomes turbulent near the entrance where the pipe is already straight, a relation expressing the momentum thickness in terms of skin friction coefficient and a shape parameter is established by means of the momentum integral and equation of continuity. With the assumptions that the skin friction coefficient and shape parameter are functions of the momentum Reynolds number, a growth of the momentum thickness in the downstream direction is derived upon the choice of special form of these functions. Similar results for the pressure drop and total head loss are also presented. *Y. H. Kuo* (Peking).

Reid, W. H. On the approach to the final period of decay in isotropic turbulence according to Heisenberg's transfer theory. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 559–563.

L'auteur, définissant la période finale de la turbulence isotrope par la propriété

$$\overline{u^2} \propto (t-t_0)^{-5/2}$$

se propose d'en préciser la structure, en admettant la validité de la théorie du transfert d'Heisenberg. L'équation d'évolution de la fonction spectrale s'écrit

$$\frac{\partial E}{\partial t} = T - 2\nu k^2 E,$$

où E et T sont en outre liés par l'équation bien connue d'Heisenberg. On cherche des solutions de la forme

$$E(k, t) = \nu^{3/2}(t-t_0)^{-1/2} R_\lambda^2 F(x, R_\lambda)$$
$$T(k, t) = \nu^{3/2}(t-t_0)^{-3/2} R_\lambda^2 U(x, R_\lambda)$$

où $x = (\nu k^2 t)^{1/2}$ et où R_λ est le nombre de Reynolds de la turbulence.

On peut développer F et U en séries de la forme

$$F(x, R_\lambda) = \sum_{n=0}^{\infty} F_n(x)(\varkappa R_\lambda)^n,\ U(x, R_\lambda) = \sum_{n=1}^{\infty} U_n(x)(\varkappa R_\lambda)^n,$$

où $\varkappa$ est la constante d'Heisenberg, et calculer successivement les F_n et les U_n. On obtient en particulier les expressions de F_0 et U_1.

Le facteur de distorsion (skewness) S de $-\partial u_1/\partial x_1$

tend vers $0{,}5769\varkappa$ lorsque $t\to\infty$. Entre la période initiale et la période finale, S varie dans un rapport de 1 à 3 environ, quelle que soit la valeur de R_λ. *J. Bass* (Paris).

Gifford, Frank, Jr. The relation between space and time correlations in the atmosphere. J. Meteorol. 13 (1956), 289–294.

Using the interpolation formula for the energy spectrum

$$F\left(\frac{k}{k_0}\right)=\frac{2}{3}\left(\frac{k}{k_0}\right)\left[1+\left(\frac{k}{k_0}\right)^2\right]^{-4/3}$$

due to Ogura [J. Meteorol. Soc. Japan (2) **31** (1953), 355–369] the author evaluates the Eulerian time correlation for a one-dimensional turbulent velocity field with an arbitrary level of turbulence. Both large- and small-scale fluctuations are then considered, and the results compared with observations. *M. H. Rogers.*

Pan, L. J.; and Kuo, Y. H. Compressible viscous flow past a wedge moving at hypersonic speeds. J. Math. Phys. **35** (1956), 179–193.

Les auteurs étudient l'écoulement hypersonique autour d'un dièdre éffilé. La viscosité et la conductivité thermique sont prises en considération; le mouvement du fluide est étudié d'une part loin de l'obstacle: champ extérieur, d'autre part au voisinage de l'obstacle: couche limite. Dans les deux cas la solution est développée suivant les puissances d'un paramètre ε inversement proportionnel à la racine carrée du nombre de Reynolds. La première approximation (termes en $\sqrt{\varepsilon}$ ou en ε, suivant les cas) est determinée de façon explicite; dans le champ extérieur une forme asymptotique est obtenue pour la seconde approximation (termes en $\varepsilon^{3/2}$). La discussion fait apparaître les résultats suivants: les effets dûs aux ondes réfléchies sur le premier choc sont faibles; la variation d'entropie dans la couche froide engendre une grande augmentation de température et de faibles pressions négatives; le théorie des ondes simples permet une bonne détermination de la pression sur l'obstacle ainsi que de la composante tangentielle de la force de frottement.

H. Cabannes (Québec).

★ **Clutterham, D. R.; and Taub, A. H. Numerical results on the shock configuration in Mach reflection.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 45–58. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The authors carry out detailed calculations, based on certain simplifying assumptions, giving the magnitude of an angle occuring in Mach reflection of shock waves from a rigid wall. They find discrepancies to exist between the theoretical results and experimental values.

H. Polachek (Carderock, Md.).

Černyĭ, G. G. Gas flow past bodies at high supersonic speed. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 221–224. (Russian)

Let L be the boundary of a plane cross section of a body in a steady plane or axisymmetric hypersonic flow. Let P be any point in the fluid in the plane of L, and R the foot of the perpendicular from P to L. Introduce curvilinear coordinates x, y where x is arclength from some reference point along L to R and $y=RP$. The author seeks y, velocity components u, v, pressure p, and density ϱ as functions of x and the stream function ψ. He sets $\varrho=(\varrho_0/\varepsilon)+\varrho_1$ and expresses the other dependent variables in the form $y=y_0+\varepsilon y_1$, etc. where $\varepsilon=(\gamma-1)/(\gamma+1)$, and determines y_0, y_1 etc. so that the equations of motion, boundary conditions at L, and shock conditions are satisfied to terms of order ε. Explicit forms are given for a plane (circular cylindrical) duct with wedge-shaped (truncated conical) walls. Approximate drag coefficients obtained therefrom for the limiting cases of ductless wedge or cone agree closely with the exact values for $M=\infty$ over a wide range of cone or wedge half angles.

J. Giese (Aberdeen, Md.).

Černyĭ, G. G. One-dimensional unsteady motion of a perfect gas with strong shock waves. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 657–660. (Russian)

The method of the preceding paper has been applied to the unsteady flow field behind plane, cylindrically, or spherically symmetrical shocks advancing into stagnant gas. In terms of shock location $R_0(t)$ general forms have been found for the zeroth and first order coefficient functions which satisfy the Lagrangean equations of motion to terms of order ε and the shock conditions simplified by retaining only the highest order terms in shock velocity. An approximate method to determine $R_0(t)$ is described for flows produced by a ν-dimensional expanding piston when the distance $\bar{R}(t)$ from the origin to the surface of the piston is prescribed. Explicit forms of the coefficient functions have been found for $\bar{R}(t)=Ct^{n+1}/(n+1)$ where C is a constant. Approximate formulas obtained therefrom for the pressure at the piston agree well with numerical results obtained for an exact theory for $\nu=3$, $0.6\leqq n\leqq 1$. *J. Giese* (Aberdeen, Md.).

★ **Tsien, H. S. The Poincaré-Lighthill-Kuo method.** Have
Advances in applied mechanics, vol. IV, pp. 281–349. Academic Press Inc., New York, N. Y., 1956. $10.00.

The method referred to in the title was originally proposed and apllied by M. J. Lighthill [Phil. Mag. (7) **40** (1949); 1179–1201; Aero. Quart. 3 (1951), 193–210; MR **11**, 518, **13**, 699]. It deals with differential equation problems involving a parameter ε in such a way that the solution has, for $\varepsilon=0$, a singularity in the region of interest. Poincaré's classical perturbation techniques are then inappliable in regions containing that singularity. Lighthill showed that in many cases this difficulty can be overcome by introducing a new independent variable ξ related to the old variable x by a series of the form $x=\xi+\sum_{n=1}^{\infty}x_n(\xi)\varepsilon^n$, and by determining the $x_n(\xi)$ by a suitable recursion scheme. In the case of partial differential equations more than one independent variable may have to be so transformed. Y. H. Kuo treated some problems of viscous flow successfully with the help of this idea by combining it with the "stretching" transformations employed in boundary layer theory [J. Math. Phys. **32** (1953), 83–101; MR **15**, 999]. The author gives a connected account of this method, including also applications by G. B. Whitham, C. C. Lin and Ph. Fox. The arguments are largely heuristic and require non-trivial modifications in each application. {It seems to this reviewer that Poincaré's connection with this technique is so indirect that he should not be mentioned in the name of the method. For if the name of every method were to include all its important intellectual ancestors mathematical terminology would become impossibly cumbersome.}

W. Wasow (Madison, Wis.).

Pivko, Svetopolk. Pressure distribution and aerodynamic properties of high-speed aircraft. A method for determining the main aerodynamic properties of high-speed aeroplanes. Aircraft Engrg. **28** (1956), 259–261.

A method for determination of the velocity or pressure distribution and the aerodynamic properties of a low-aspect-ratio swept wing with slender body of elliptical cross-section and vertical tail surface, having arbitrary form, is briefly presented.

The method can be used for the prediction of the load distribution, aerodynamic forces and moments on inclined slender wing, body and vertical tail combinations travelling at subsonic or supersonic speeds. (Author's summary.) *R. M. Morris* (Cardiff).

Plesset, M. S.; and Mitchell, T. P. On the stability of the spherical shape of a vapor cavity in a liquid. Quart. Appl. Math. **13** (1956), 419–430.

This paper deals with the perturbation equation governing the stability of a vapor-filled spherical cavity in a non-viscous liquid. Explicit solutions of the perturbation equation are given in the following cases: (i) Expanding cavity, no surface tension; (ii) collapsing cavity, no surface tension; (iii) expanding cavity with surface tension; (iv) cavity collapse under surface tension alone. (In each case the solution involves a certain hypergeometric function.) From an examination of the explicit solutions, the authors find that the expanding cavity with surface tension is stable, but that when surface tension is negligible then needlelike irregularities in the spherical interface may grow to significant amplitude. A collapsing cavity, on the other hand, is unstable as the radius R tends to zero [Birkhoff's theorem; cf. Quart. Appl. Math. **12** (1954), 306–309; MR **16**, 413; see also the paper reviewed below]. In spite of this instability, it is found that the distortion amplitudes remain small over a range

$$1 \geq R/R_0 \gtrsim 0.2,$$

where R_0 is the initial radius, and where the value 0.2 is, of course, approximate.

J. B. Serrin (Minneapolis, Minn.).

Birkhoff, Garrett. Stability of spherical bubbles. Quart. Appl. Math. **13** (1956), 451–453.

The perturbation equation for a spherical vapor-filled bubble can be written in the form

$$x''+p(t)x'+q(t)x=0, \tag{1}$$

where $x(t)$ is a perturbation amplitude and p and q are known functions. This note contains the following general criterion for the stability of solutions of (1): If $q<0$, or if $q>0$ and $2pq+q'<0$, then (1) is unstable; if $q>0$ and $2pq+q'>0$, then (1) is stable. More specifically, if $q<0$ it is shown that $x(t)$ grows forever in magnitude as $t\to\infty$, while if $q>0$ the solution oscillates with increasing or decreasing amplitude depending on whether $2pq+q'$ is less or greater than zero. This result is applied to obtain in a different way the author's earlier conclusion that collapsing vapor-filled bubbles are unstable [cf. Quart. Appl. Math. **12** (1954), 306–309; MR **16**, 413].

J. B. Serrin (Minneapolis, Minn.).

Betz, Albert. Näherungsformeln für die Zirkulationsverteilung um eng stehende Schaufeln von Strömungsgittern. Z. Flugwiss. **4** (1956), 166–169.

Taylor, G. I.; and Miller, J. C. P. Fluid flow between porous rollers. Quart. J. Mech. Appl. Math. **9** (1956), 129–135.

This paper derives and solves a differential equation for the pressure distribution in a viscous fluid passing the contact point region between a pair of porous rollers. Porosity is explored because the formula for the case of impervious rollers with a finite gap between them gives divergent pressures at the contact region when the gap is made to approach zero. The treatment has mathematical interest because of the elegant solution in closed form. The rejection of a direct solution in descending powers of the independent variable x_1 is not clear since the authors use this expansion derived indirectly from the closed form solution for numerical estimates of the pressure. The directly derived expansion has the required number of arbitrary constants, one of which is determined by the zero pressure condition at infinity while the other has only one admissible value, yielding a finite solution at the roller contact point. The one admissible value may be approximated by noting in the differential equation that the pressure gradient approximates one near the contact point. It does seem that a solution of the physical problem with a finite pressure jump at the contact point is also plausible. {Since the derived differential equation admits only solutions which are zero or infinite at this point, one feels that the problem has not been completely solved. A solution of the porous roller problem with a gap between them which could be made to approach zero would thus seem desirable.} *M. G. Scherberg.*

Craven, Arthur H. A potential flow model for the flow about a nacelle with jet. Coll. Aero. Cranfield. Rep. no. **101** (1956), 46 pp. (8 plates).

Craven, Arthur H. The free streamline method applied to the flow at the rear of a duct. Coll. Aero. Cranfield. Rep. no. **99** (1956), 19 pp. (5 plates).

Moshinsky, Marcos. Dispersion of sound waves by a drop of liquid. Univ. Nac. Autonoma Mexico. An. Inst. Fis. **1** (1955), 55–68. (Spanish. English summary)

The equations for and boundary conditions on the velocity potential of a sound wave scattered by (1) a spherical drop of liquid and (2) a soap bubble in an adiabatic gas are derived. Solutions are of the spherical harmonic type. The characteristics of the scattering are discussed for the limiting cases of high and low frequency, and the behavior of the two types of scatterers near resonance is studied. *R. N. Goss* (San Diego, Calif.).

Gazaryan, Yu. L. On the guided propagation of sound waves in an inhomogenous medium. Akust. Ž. **2** (1956), 133–136. (Russian)

The author discusses the acoustic field of a point source in a channel in which the sound velocity varies according to a law proposed originally for the electromagnetic case by P. S. Epstein [Proc. Nat. Acad. Sci. U.S.A. **16** (1930), no. 10, 627–637], namely

$$\varepsilon^2(z)=\left[\varepsilon_0^2-\frac{1}{2}(\varepsilon_+^2+\varepsilon_-^2)\right]\operatorname{sech}^2(z/h)+\frac{1}{2}(\varepsilon_+^2-\varepsilon_-^2)\tanh(z/h)+\frac{1}{2}(\varepsilon_+^2+\varepsilon_-^2),$$

where $\varepsilon(z)=1/c(z)$, $\varepsilon_0=\varepsilon(0)$, $\varepsilon_+=\varepsilon(+\infty)$, $\varepsilon_-=\varepsilon(-\infty)$, and h is the depth of the channel. *R. N. Goss.*

Černov, L. A. Correlational properties of a wave in a medium with random inhomogeneities. Akust. Ž. **2** (1956), 211–216. (Russian)

In a previous paper [Akust. Ž. **1** (1955), 89–96; MR **17**, 1252] the author has studied the correlation between fluctuations of amplitude (and phase) at two reception points due to the scattering of a sound wave by inhomogeneities randomly distributed throughout the medium. This work was limited to the case when the two receivers lie in a plane perpendicular to the direction of propagation (transverse autocorrelation). In the present paper the receivers are both placed along the line of propagation (longitudinal autocorrelation). It is assumed, as in the earlier work, that the random variations from the mean of the index of refraction are small relative to the mean, that they are large-scale relative to a wavelength, and that the spatial distribution of the inhomogeneities is quasi-static. The author first obtains general expressions for the longitudinal space autocorrelations for amplitude and phase fluctuations in terms of the correlation coefficient N of the index of refraction. For long ranges from the source and for a Gaussian form of N, he reduces these to a simple form. He shows that longitudinal correlation extends over a greater distance than transverse. Finally he determines the time autocorrelation functions for amplitude and phase and uses his result to explain a noteworthy variability discovered by D. Mintzer [J. Acoust. Soc. Amer. **26** (1954), 186–190; MR **15**, 1002] in correlation functions computed from certain experimental data. *R. N. Goss.*

Brehovskih, L. M. On focusing of sound waves by inhomogeneous media. Akust. Ž. **2** (1956), 124–132. (Russian)

The radiation from a point source in the half-space $z>0$ occupied by a homogeneous medium is observed at another point within the same medium. For $z<0$ the wave number and density are slowly varying functions of z. The sound field is represented in integral form as the superposition of plane waves, in which integral the plane-wave reflection coefficient appears as a function of wave number. Under the assumption of sufficiently high frequency this coefficient is computed by the WKB method. The resulting integral is treated by the saddle-point method to yield the ray approximation for the field. The condition for the rays to have a focus is brought out, and the field in the vicinity of the caustics is studied. An example is given. The procedure is then extended to the case in which the source is situated in the plane $z=0$.

R. N. Goss (San Diego, Calif.).

Isakovič, M. A. On scattering and emission of waves by statistically inhomogeneous and statistically oscillating surfaces. Akust. Ž. **2** (1956), 146–149. (Russian)

The author applies to two new problems the method used in an earlier paper [Ž. Eksper. Teoret. Fiz. **23** (1952), 305–314; MR **14**, 700] to treat the scattering of waves from a statistically rough surface. In the first he allows the coefficient of reflection to be a random function of position in the x-direction on the surface, and in the second he assumes that on the surface the normal velocity component of the field varies randomly with x. The chief simplifying assumption is that the scattering surface is locally plane. In each case the average intensity of the Fraunhofer field is obtained in terms of an integral containing the correlation coefficient $r(\tau)=r(x-x')$ of the random property, and the solution is carried out completely for the function $r(\tau)\exp(|\tau|/\tau_0)$. *R. N. Goss.*

See also: Buchner, p. 120; Krasil'nikov and Obuhov, p. 154; Kopzon, p. 163; Melvin and Edwards, p. 164; Kopzon, p. 165; Taub, p. 177.

Optics, Electromagnetic Theory, Circuits

Shack, Roland V. Characteristics of an image-forming system. J. Res. Nat. Bur. Standards **56** (1956), 245–260.

A very clear exposition of the analysis of optical systems in terms of the image of a point object and/or the Fourier transform of this image. Examples of aberration — free systems with a circular aperture in monochromatic light are discussed and related to various experimental results. *G. L. Walker* (Southbridge, Mass.).

Köhler, H. Zur Abbildungstheorie anamorphotischer Systeme. Optik **13** (1956), 145–157.

A discussion of the paraxial and third order image theory of anamorphic lens system (i.e. systems symmetric in two orthogonal planes) with a summary of previous investigations of this problem. These systems are currently applied in some wide screen movie processes.

G. L. Walker (Southbridge, Mass.).

Kogan, S. Ya. On the method of spherical functions in atmospherical optics. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1053–1055. (Russian)

The author considers the Boltzman integro-differential equation in connection with the problem of the isotropic scattering of light through the atmosphere. The same equation with the same boundary conditions was previously treated by Chandrasekhar [Astrophys. J. **99** (1944), 180–190; MR **6**, 76] and also by Wang and Guth [Phys. Rev. (2) **84** (1951), 1092–1111; MR **14**, 503] by expanding the unknown distribution function in a series of Legendre polynomials. However, in the latter papers there appeared to be an ambiguity both in the manner of defining successive approximations to the solution of the equation and in the manner of introducing corresponding approximate boundary conditions. In the present paper it is shown how the latter ambiguities can be removed and a unique solution obtained. *J. F. Heyda.*

Fedorov, F. I. On the theory of total reflection. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 465–468. (Russian)

This paper treats the general case of total reflection for arbitrary elliptical polarization. The results reveal certain basic aspects of total reflection, which are not evident in the special treatments dealing with light linearly polarized either parallel or perpendicular to the plane of incidence. According to the author, the failure to consider the general case resulted in errors in an article by Wiegrefe [Ann. Physik (6) **45** (1914), 465–480]. Furthermore he finds erroneous the statement in Born's Optik [Springer, Berlin, 1933, § 13] to the effect that in the case of total reflection the flow of energy in the second medium is parallel to the plane of incidence. *J. E. Rosenthal.*

Smythe, W. R. Charged right circular cylinder. J. Appl. Phys. **27** (1956), 917–920.

The charge density on a conducting cylindrical can is

found by a series technique. The ends are described by planes $z=\pm c$ and the cylindrical surface by $\varrho=a$. Symmetry indicates that the density function σ_e on the ends is even in ϱ, and σ_s on the curved surface is even in Z. Near the edge the charge density is assumed, by analogy with the right-angled wedge, to be proportional to $\delta^{-\frac{1}{2}}$ where δ is the distance from the edge. These considerations lead to the postulation of a charge distribution of the form

$$\sigma_s=\sum_{n=0}^{\infty}A_n[(c^2-z^2)/a^2]^{n-\frac{1}{2}},\quad \sigma_e=\sum_{n=0}^{\infty}B_n[1-(\varrho/a)^2]^{n-\frac{1}{2}}.$$

The coefficients are determined explicitly in terms of hypergeometric functions through the conditions that the two densities be equal on the edge and that the potential be constant in the interior of the conductor. Computed values of the coefficients for $n=0$, 1, 2 are given for ratios $c/a=\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4. Capacitance also is computed. Possible application of the method to other axially symmetric problems is mentioned. *A. A. Blank.*

Galasiewicz, Z. Generalization of the method of supplementary variables to systems composed of two kinds of particles. Acta Phys. Polon. **15** (1956), 49–62. (Russian summary)

Schon in einer früheren Arbeit [Acta Phys. Polon. **14** (1955), 373–375; MR **17**, 1159] hat der Verfasser die Berechnungen von D. N. Zubarev [Ž. Eksper. Teoret. Fiz. **25** (1953), 548–559] bezüglich der Bewegung eines Plasmas auf den Fall von verschiedenen Massen und Ladungen (Elektronen und Ionen) erweitert. (Bekanntermeise stimmen die Berechnungen von Zubarev ihrem physikalischen Inhalt nach mit denen von D. Bohm und D. Pines [Phys. Rev. (2) **82** (1951), 625–634; **85** (1952), 338–353; **92** (1953), 609–625; MR **12**, 886], überein.) Vorliegende Arbeit ist eine Erweiterung der zitierten. Der Verfasser geht aus der auch die Wechselwirkung der Elektronen und Ionen enthaltenden Hamiltonfunktion aus, die er mit Hilfe gewisser Umformungen und der unitären Transformation $U=\exp(-ih^{-1}\sum_{k,k\geq k_0}\mathbf{P}_k\varrho_k)$ (wo $\mathbf{P}_k=-ih\,\partial/\partial\mathbf{Q}_k$ ist und mit der auch die Zubarevschen Nebenbedingungen transformiert werden) in solch eine Form bringt, dass man die auftretenden Bewegungen in eine kollektive Bewegung des Plasmas und eine Bewegung die von den Coulombschen Wechselwirkungen kurzer Reichweite herrührt, zerlegen kann. Zwischen denen treten jedoch auch Wechselwirkungsglieder auf. Erstere Bewegungsart nennt er die Anregungen von Bosonen-Typ und die letztere von Fermionen-Typ. Der ersteren Bewegung entspricht eine Schwingung mit der Frequenz $\sqrt{(\omega_1^2+\omega_2^2)}$, wo ω_1 die Langmuirsche Frequenz der Elektronen und ω_2 die der Ionen bedeutet. Im Falle einer verschwindenden Wechselwirkung treten ω_1 und ω_2 voneinander unabhängig auf. Die Differenz der in diesen zwei Fällen auftretenden Nullpunktsenergien ermöglicht die Abschätzung der Bindungsenergien einiger Alkalimetalle, die selbstverständlich nicht so gute Resultate liefern kann, wie die Wigner-Seitzsche Methode.

Th. Neugebauer (Budapest).

Galasiewicz, Z. On the equivalence of the Zubarev method and the Bohm-Pines method for systems of two types of particles. Acta Phys. Polon. **15** (1956), 79–87. (Russian summary)

In dieser, gleichzeitig mit der oben besprochenen Arbeit veröffentlichten, zeigt der Verfasser, dass die auf zwei Teilchensorten (Elektronen und Ionen) verallgemeinerte Methode von D. N. Zubarev [Ž. Eksper. Teoret. Fiz. **25** (1953), 548–559] und die ebenso verallgemeinerte von D. Bohm und D. Pines [Phys. Rev. (2) **82** (1951), 625–634; **85** (1952), 338–353; **92** (1953), 609–625; MR **12**, 886] einander vollständig äquivalent sind. Im ersten Teil der Arbeit fasst der Verfasser seine Resultate bezüglich der Theorie von Zubarev zusammen, im zweiten verallgemeinert er die Resultate von Bohm und Pines und in der dritten zeigt er die Äquivalenz der beiden, in dem er durch eine Transformation die von ihm angegebene Hamiltonsche Funktion in die verallgemeinerte von Bohm und Pines überführt. Von dieser Transformation wird noch gezeigt, dass sie die Heisenbergschen Vertauschungsrelationen unberührt lässt und das beweist, dass sie in eine quantenmechanischen Sinne kanonische Transformation ist. Die analytische Form dieser unitären Transformation wird angegeben. *Th. Neugebauer* (Budapest).

Wilhelmsson, Hans. On the reflection of electromagnetic waves from a dielectric cylinder. Chalmers Tekn. Högsk. Handl. no. **168** (1955), 17 pp.

The problem solved is that of the reflection of an arbitrarily polarized plane electromagnetic wave by an infinite dielectric cylinder. The general case in which the direction of propagation is not perpendicular to the axis of the cylinder is considered. The method is basically that given in M. Born [Optik, Springer, Berlin, 1933] for the dielectric sphere. There is however, the added complication introduced by a direction of propagation which does not fit handily into the coordinate system. Calculations are limited to sizes of cylinders for which the resultant series of Bessel functions are rapidly convergent.

W. K. Saunders (Washington, D.C.).

Karp, S. N.; and Williams, W. Elwyn. Equivalence relations in diffraction theory. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-83** (1955), i+16 pp.

Certain equivalence relationships between different problems occurring in diffraction theory are established. The method used is essentially Schwartz principle of analytic continuation by reflection across a straight boundary. Some explicit examples are drawn from diffraction theory. (Author's summary.) *A. E. Heins.*

Piddington, J. H. The four possible waves in a magneto-ionic medium. Phil. Mag. (7) **46** (1955), 1037–1050.

It is pointed out that, in an atmosphere of ions and electrons in a constant magnetic field, there are, apart from the two known magneto-ionic waves, two more waves. One is a 'magnetic sound' wave, due to the motion of the ions, and is not studied here. The other is a 'magnetic plasma' wave in the plasma of electrons, and is present even at radio frequencies. The dispersion equation is derived, taking into account the electron gas pressure while the ions only serve to provide the constant neutralizing background. The properties of the plasma wave are studied for various directions of propagation with respect to the magnetic field. The occurrence of such waves in the solar atmosphere is discussed, and a mechanism is suggested by which they may give rise to solar radio noise. *N. G. van Kampen* (Utrecht).

★**Brillouin, Léon; et Parodi, Maurice. Propagation des ondes dans les milieux périodiques.** Masson et Cie, Paris; Dunod, Paris, 1956. iv+347 pp. 4600 francs.

This is a translated and augmented edition of the senior

author's "Wave propagation in periodic structures" [McGraw-Hill, New York, 1946; MR 8, 422; 9, 735]. The four additional chapters written by M. Parodi contain material on limit problems and finite periodic structures; use of gegenbauer polynomials in the design of filters; slow wave guide structures; motion of the electron in a moving periodic field. The new material is a valuable addition to what has proven to be a useful and popular treatise. It is to be hoped that should the authors publish a second English edition they will see fit to prepare one further chapter on the applications of their methods to the important field of the traveling wave tube and the backward wave oscillator. It seems perverse that they have built such elaborate and elegant machinery and have then not used it for the solution of these current problems.

W. K. Saunders (Washington, D.C.).

Darlington, Sidney. A survey of network realization techniques. Bell Tel. System Monograph **2620** (1956), 7 pp.

Bordewijk, J. L. Inter-reciprocity applied to electrical networks. Appl. Sci. Res. B. **6** (1956), 1–74.

A concept called inter-reciprocity is introduced which is an extension of the concept of reciprocity. Transposing the impedance matrix of a $2n$-pole gives the impedance matrix of another $2n$-pole; the two networks are inter-reciprocal. The topological process corresponding to this transposition is analyzed. Application is made to transmission and noise problems of triode and transistor networks. *R. J. Duffin* (Pittsburgh, Pa.).

★ **Papoulis, A. Network response in terms of behavior at imaginary frequencies.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 403–424. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

An inversion formula for the Laplace transform is developed which employs the value of the transform at a sequence of points in arithmetic progression on the real axis. The relation to the classical moment problem is indicated. *R. J. Duffin* (Pittsburgh, Pa.).

★ **Klinkhamer, J. F. Reflectionless transmission through $2n$-terminal-pair networks.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 361–384. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

The author defines and systematically analyzes an image impedance matrix and a propagation constant matrix for a chain of $4n$-poles. For $n=1$ this reduces to the classical case of the scalar image impedance and propagation constant used for long lines and wave filters.

R. J. Duffin (Pittsburgh, Pa.).

★ **Belevitch, V. Synthesis of four-wire conference networks and related problems.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 175–195. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

The scattering matrix S of an ideal transformer network is directly related to an orthogonal matrix N which gives the turns-ratios of the transformers. In particular, conference networks lead to the study of orthogonal matrices N whose elements are 0 or $\pm s$ for a real constant s. The synthesis of such networks is aided by symmetry considerations. *R. J. Duffin* (Pittsburgh, Pa.).

★ **Oono, Y. On pseudo-scattering matrices.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 99–118. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

A matrix termed a pseudo-scattering matrix is introduced and studied. This matrix may be of value in determining the minimum number of ideal gyrators sufficient to synthesize a non-reciprocal network with n pairs of terminals. *R. J. Duffin* (Pittsburgh, Pa.).

★ **Carlin, H. J. Synthesis of nonreciprocal networks.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 11–44. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

A description is given of ideal gyrators and their practical realization. The synthesis of various types of circuits by use of gyrators is described.

R. J. Duffin.

★ **Troitskii, V. N. Ultra-shortwave propagation over long distances beyond the limits of the horizon.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 26 pp.

Translated from Radiotehnika **11**, no. 5, 1956, pp. 3–20.

★ **Solov'ev, E. G.; and Belous, L. V. To the theory of a spiral line surrounded by a cylindrical semiconducting shell.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 7 pp.

Translated from Radiotehnika **11**, no. 4, 1956, pp. 31–35.

★ **Solov'ev, E. G. Electromagnetic wave propagation between two circular cylindrical surfaces in the presence of longitudinal diaphragms situated periodically.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from Radiotehnika **11**, no. 1, 1956, pp. 57–60.

★ **Rytov, S. M. Theory of electric fluctuations and thermal radiation.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 8 pp.

Translated from P. N. Lebedev Phys. Inst., Akad. Nauk Press, Moscow, 1953.

Isabeau, J. L'utilisation de la fonction de Dirac dans l'analyse spectrale à deux dimensions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 840–853.

En se basant sur les propriétés, énoncées et démontrées, de la fonction impulsion de Dirac généralisée et étendue à un espace à deux dimensions, l'auteur fait l'étude spectrale des phénomènes de moirages, c'est-à-dire des figures obtenues lorsque, dans un tissu, l'épaisseur du fil, sa couleur ou toute autre donnée physique varie au cours du tissage.

Cette étude est applicable également à l'analyse des figures de moirages obtenues sur les écrans de télévision.

Résumé de l'auteur.

See also: Helman, p. 125; Meecham, p. 135; Krasil'nikov and Obuhov, p. 154; Bunkin, p. 160; Gazaryan, p. 168; Isakovič, p. 169; Cahen, p. 177; Radzievskiĭ, p. 179; Pöschl, p. 181; Schober, p. 181.

Thermodynamics and Heat

Chmelka, Fritz. Wärmespannungen in einem Prandtl-Reussschen Körper. Österreich. Ing.-Arch. **10** (1956), 133–140.

Gordov, A. N. On the propagation of temperature waves in a cylinder. Ž. Tehn. Fiz. **25** (1955), 1444–1448. (Russian)

Paper gives an exact solution to the trivial problem of temperature variation inside a solid infinite cylinder of radius R outside of which the temperature t fluctuates harmonically about a mean value t_0 with an amplitude A_e. The resistance to the transfer of heat from the external fluid medium to the cylinder is allowed for in the usual way by stipulating a constant film coefficient of heat transfer, α.

The steady-state solution is then averaged over the volume of the cylinder, giving a closed expression for the mean cylinder temperature, $\bar{\theta}_V$, at large times. The latter varies harmonically with time, the amplitude of the ratio $(\bar{\theta}_v - t_0)/A_e$ being $A_V(hR, HR)$, and the phase angle φ_V being a function of the same magnitudes. The function A_V is tabulated (hR=0.1, 0.5, 1.0, 2.0, 3.0, 5.0, 10.0, ∞ and HR=0.5, 1—(1)—10). Here hR is the Nusselt number $\alpha R/\lambda$ formed with the conductivity, λ, of the solid, and $HR=R\sqrt{(\omega/a)}$ (ω - circular frequency of external temperature fluctuation, a — thermal diffusivity of solid). The function φ_V is not tabulated.

A comparison with an earlier approximate formula due to Kondratev is also given.

A solution to the same problem is given in Carslaw and Jaeger [Conduction of heat in solids, Oxford, 1947, p. 276; MR **9**, 188]. The problem was originally solved by Gröber [Z. Verein. Deutsch. Ing. **70** (1926), 1266–1268] who provided graphical aids for numerical solutions.

J. Kestin (Providence, R.I.).

Chao, C. C.; and Weiner, J. H. Heat conduction in semi-infinite solid in contact with linearly increasing mass of fluid. Quart. Appl. Math. **14** (1956), 214–217.

Paper contains an interesting, if somewhat academic, variant on the problem of one-dimensional, time-dependent heat conduction. It treats the case of a semi-infinite solid of infinitely large cross-sectional area in contact with an increasing mass of fluid ("poured" at constant rate) of initially constant temperature. During the pouring process the instantaneous mass of the liquid is assumed to have instantly acquired the temperature of the solid surface (infinitely large thermal diffusivity of liquid).

Under these assumptions it is possible to obtain a solution in closed form: it contains a complementary error-function term and a definite integral. The mathematical interest resides in the fact that the Fourier equation is solved subject to a boundary condition expressed in terms of a partial differential equation. The method of solution skilfully applies the Laplace transform technique. A graph of the variation of the ratio u/U of instantaneous fluid temperature to original fluid temperature in terms of the reduced time t/s^2 is given in the paper. Here $s=Kmcx^{\frac{1}{2}}$ (K — thermal conductivity of solid, m — constant mass rate of pouring, c — specific heat of liquid, x — thermal diffusivity of solid). *J. Kestin.*

See also: Truesdell, p. 162; Bass, p. 166.

Quantum Mechanics

Koppe, H. Zur Definition der Spinrichtung in der Diracschen Theorie. Z. Physik **145** (1956), 398–402.

The author discusses a variety of operators which commute with the Dirac Hamiltonian and which depend linearly on the spin matrices and the components of an arbitrary unit vector. *A. H. Taub* (Urbana, Ill.).

Feenberg, Eugene. Invariance property of the Brillouin-Wigner perturbation series. Phys. Rev. (2) **103** (1956), 1116–1119.

In a previous paper [Goldhammer and Feenberg, Phys. Rev. (2) **101** (1956), 1233–1234] a refinement of the Brillouin-Wigner perturbation method was discussed. The improvement involved inserting into the B-W wave functions a set of constants G_i, where $G_i=1$ for the usual B-W results. The energy was then minimized with respect to the G_i according to a variational principle in order to obtain a best solution. In the present paper the author considers the situation where one adds to the unperturbed hamiltonian, H_0, a contribution $(\mu-1)(H_0-E)$ (subtracting this quantity from the potential) in order to represent a reduced mass effect. It is then shown that the procedure of minimizing with respect to the G_i leaving the value of μ fixed gives the same result as setting the G_i to unity and minimizing via the variational principle with respect to μ. The optimum displacement of the zeroth order energy spectrum is also discussed. *R. Arnowitt.*

Kamefuchi, Susumu; and Tanaka, Shô. On the Jauch field. Progr. Theoret. Phys. **14** (1955), 225–242.

Jauch [Helv. Phys. Acta. **27** (1954), 89–98; MR **16**, 101] has proposed the following anti-commutation relations for a spinor field,

$$\{\psi(x), \psi^*(x')\}=\delta^3(x-x'), \{\psi(x), \psi(x')\}=\varrho\delta^3(x-x') \quad (0\leq\varrho\leq 1).$$

For $\varrho=0$ and 1, the field reduces respectively to a Dirac or a Majorana field. Jauch had concluded that different values of ϱ give fields which are not unitary equivalent and that except for the case $\varrho=0$, all these fields are necessarily neutral.

The authors consider a free Lagrangian for the Jauch field, from which the anti-commutation relations above follow by usual prescriptions of field-quantization. For $\varrho\neq 1$, this Lagrangian is equivalent to a Dirac-field Lagrangian.

Thus contrary to Jauch's result all such fields can be charged Dirac fields. This conclusion is further substantiated by introducing gauge invariant interactions.

A. Salam (Cambridge, England).

Tomonaga, Sin-itiro. Elementary theory of quantum-mechanical collective motion of particles. I, II. Progr. Theoret. Phys. **13** (1955), 467–481, 482–496.

This is an "elementary theory" of quantum-mechanical collective motion of particles which in author's phrase, "contains nothing highbrow such as second quantization, canonical transformation and so on." The method is a natural generalization of the use of centre of mass coordinates to describe translational motions and to separate them from internal motions of the system. To illustrate, consider surface oscillations of an incompressible system consisting of N particles. Assume that the motion is 2-dimensional and irrotational, described

by a displacement potiential $\phi=\frac{1}{2}(x^2-y^2)$. The ideal description of the system would be in terms of collective momentum and displacement co-ordinates π and ξ, and internal coordinates ζ, which should satisfy,

$$(1)\quad [\pi,\xi]=i\hbar,\ [\pi,\zeta]=0,\ \sum_n \frac{\partial\xi}{\partial x_n}\frac{\partial\zeta}{\partial x_n}+\frac{\partial\xi}{\partial y_n}\frac{\partial\zeta}{\partial y_n}=0.$$

The last condition guarantees that internal motion can be treated separately from collective motion.

With the representation

$$-i\hbar\sum\left(x_n\frac{\partial}{\partial x_n}-y_n\frac{\partial}{\partial y_n}\right),$$

for π, the "natural choice" for ξ seems to be

$$\xi=\frac{1}{N}\sum(\log x_n-\log y_n),$$

since this gives $[\pi,\xi]=-i\hbar$. However it is no longer possible to find internal co-ordinates ζ, satisfying (1).

The author suggests using for the collective displacement coordinate ξ the expression

$$\xi=2(NR_0^2)^{-1}\sum_n\phi(x_n,y_n).$$

The difficulties connected with the definition of internal coordinates ζ (eq. (1)) no longer arise. However, now $[\pi,\xi]=-i\hbar\,2(NR_0^2)^{-1}\sum_n(x_n^2+y_n^2)$. For large N,

$$\sum_n(x_n^2+y_n^2)$$

can be replaced by its mean value $N\langle r^2\rangle$, so that by choosing $R_0^2=2\langle r^2\rangle$, the canonical commutation relation is approximately satisfied (i.e. $1/\sqrt{N}$ is neglected against unity).

The construction and exploitation of such quantities ξ forms the essence of the author's method. In the second paper the method is applied to longitudinal oscillations of particles interacting through Coulomb forces. The results of previous workers [e.g. Bohm and Pines, Phys. Rev. (2) **92** (1953), 609–625] are reproduced. *A. Salam.*

Gürsey, Feza. On a conform-invariant spinor wave equation. Nuovo Cimento (10) **3** (1956), 988–1006.

The author shows that the nonlinear equation

$$(*)\qquad \gamma^\mu\partial_\mu\psi+\lambda(\bar\psi\psi)^4\psi=0$$

is invariant under the conformal group in Minkowski space-time. In (*) the γ^μ are the Dirac matrices, $\bar\psi$ is the adjoint spinor to ψ and λ is a fundamental constant. The quantity $\bar\psi\psi$ is normalized to have the dimensions of an inverse cube of a length. The proof used involves a 2×2 matrix formalism for representing both four component spinors and vectors in the Minkowski space.

A. H. Taub (Urbana, Ill.).

Takabayasi, T. Hydrodynamical description of the Dirac equation. Nuovo Cimento (10) **3** (1956), 233–241.

The author discusses equations satisfied by the scalars, vectors, pseudoscalars and pseudo-vectors defined by a four component spinor field satisfying the Dirac equation with an external electromagnetic field present. The quantities discussed are bilinear in the spinor field and its adjoint field or bilinear in the derivatives of the spinor field and the adjoint field. The equations considered are said to be equivalent to the Dirac equation and are used to give a classical interpretation of the vector and scalar quantities treated. *A. H. Taub* (Urbana, Ill.).

Takabayasi, T. New classical spin theory as the limit of the Dirac equation. Nuovo Cimento (10) **3** (1956), 242–245.

The author discusses the equations given in the paper reviewed above in the limiting case $\hbar\to0$ and compares his work with that of previous authors. *A. H. Taub.*

Bopp, Fritz. Das Korrespondenzprinzip bei korpuskularstatistischer Auffassung der Quantenmechanik. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 9–22 (1956).

The author presents six "theorems" (no proof is supplied) giving the basic structure of statistical mechanics. Four of these are mathematical, on the group properties and parameter numbers (transfinite) of the equations of motion and the density function. The other two have to do with what can be measured physically.

He appears to wish to prove two statements: (1) That only measurability changes in going from quantum to classical statistical mechanics, namely one can no longer measure both position and momentum; and (2) that these "theorems" determine the form of equations of motion of the density matrix in quantum statistical mechanics. The proof of (2) involves the solvability of a diophantine equation in transfinite numbers but is otherwise very ingenious and interesting. *P. W. Anderson.*

Jánossy, L.; und Nagy, K. Über eine Form des Einsteinschen Paradoxes der Quantentheorie. Ann. Physik (6) **17** (1956), 115–121.

The authors discuss the example of an electron scattered by a proton, the latter being in such a state that if its momentum is measured the electron scattering will show interference effects while if its position is measured no such interference will be found. This is related to the Einstein paradox [Einstein, Podolsky and Rosen, Phys. Rev. (2) **47** (1935), 777–780]. *N. Rosen* (Haifa).

Wigner, E. P. Relativistic invariance in quantum mechanics. Nuovo Cimento (10) **3** (1956), 517–532.

This is the text of a paper presented at the International Conference on Elementary Particles held at Pisa in June 1955. The general significance and consequences of relativistic invariance are discussed in so far as they concern our theoretical description of elementary particles. The standpoint of the author is that, in view of our poor knowledge of the interactions between elementary particles, invariance and symmetry properties "should be a guide in establishing the proper physical picture". Invariance considerations are also applied to a discussion of the formation or desintegration of semi-stable particles. Some comments are devoted to the question of the intrinsic parity of elementary particles. It is finally stressed that the conclusions to be reached on the basis of symmetry considerations very much depend on what is considered to be measurable in principle. *L. Van Hove.*

Dalitz, R. H.; Sundaresen, M. K.; and Bethe, H. A. A singular integral equation in the theory of meson-nucleon scattering. Proc. Cambridge Philos. Soc. **52** (1956), 251–272.

This paper is concerned with the integral equation describing an interacting meson-nucleon system in the simplest Tamm-Dancoff approximation. After obtaining an orthogonality condition relating solutions with different values of the coupling parameter, the authors show that, for given angular momentum j and given orbital angular momentum l, the equation can be reduced to a one-

dimensional integral equation in radial momentum space. Equations are also derived for the vertex function and the renormalised vertex function in the case $j=\frac{1}{2}$; these have the same kernel as the main equation, but different inhomogeneous terms.

The main aim of the paper is to obtain asymptotic expansions for the solutions of these equations when the momentum p is large. These are derived from asymptotic expressions for the kernel and expansion of the solution of the equation in a Neumann series. Since all the traces of the kernel are infinite, Fredholm determinants cannot be used; it is remarked that the existence of a Neumann series solution depends in this case on the nature of the inhomogeneous term and, even when it does exist, the associated homogeneous equation may have non-trivial (though here physically irrelevant) solutions. In particular, the equation for the vertex function (before renormalisation) has no solution.

A more detailed discussion for the case $j=\frac{1}{2}$ shows that, in the attractive case, there is a critical value of the coupling constant beyond which no physically relevant solutions exist. It is also shown that, as $p\to\infty$, the solution of the equation decreases more slowly than the Born approximation in the attractive case, and more quickly in the repulsive case. A similar discussion is outlined for other values of j.

Appendices to the paper deal with mathematical details. In Appendix B the properties of the equation are compared with those of the simpler equation

$$g(x)=\frac{b}{x}+\gamma\int_1^\infty \frac{g(y)}{\max(x,y)}\,dy,$$

whose kernel behaves in much the same way at infinity.

F. Smithies (Cambridge, England).

Pekar, S. I. Strong coupling nucleomesodynamics. I. Approximate method. Spin-charge motion. Ž. Eksper. Teoret. Fiz. **30** (1956), 304–316; supplement to **30**, no. 2, 6. (Russian. English summary)

An approximate calculation is carried out for a system consisting of an infinitely heavy extended nucleon strongly coupled to a symmetric pseudo-scalar meson field by means of a pseudo-vector (gradient) interaction. By suitable approximations the Hamiltonian is simplified and the spin-charge parts of the wave function and of the energy are obtained. *N. Rosen* (Haifa).

Baĭer, V. N.; and Pekar, S. I. Strong coupling nucleomesodynamics. II. The ground and isobar states, nucleon charge and spin. Ž. Eksper. Teoret. Fiz. **30** (1956), 317–329; supplement to **30**, no. 2, 6. (Russian. English summary)

A continuation of the work of the paper reviewed above. The wave equation of the meson field oscillations is considered. Eigenvalues of the nucleon energy, charge and spin, as well as the wave function of the system, are found for the ground and isobar states. *N. Rosen.*

Čavčanidze, V. V. On the equations of quantum electrodynamics. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 15–20. (Russian)

On the basis of certain assumptions, such as that the electromagnetic vector potential is expressed in terms of a scalar "photomeson" field, a set of field equations for the electron-positron and "photomeson" fields is obtained. *N. Rosen* (Haifa).

Lipmanov, È. M. Relativistically invariant formulation of electrodynamics without longitudinal and scalar fields. Ž. Eksper. Teoret. Fiz. **30** (1956), 583–584. (Russian)

This is a continuation of an earlier paper [same Ž. **27** (1954), 135–141; MR **16**, 318]. The field equations in the Heisenberg representation are derived from a variational principle. By means of a canonical transformation the equations of motion of the field variables are obtained in the interaction representation.

N. Rosen (Haifa).

Abrikosov, A. A. On the infrared catastrophe in quantum electrodynamics. Ž. Eksper. Teoret. Fiz. **30** (1956), 96–108. (Russian)

The infrared catastrophe is investigated by summing over diagrams. Expressions are obtained for the Green's function and for the vertex part in the approximation needed for this purpose. A generalization of the Feynman diagrams is presented in order to obtain probabilities of multiple processes. The scattering of an electron in an external field, with radiation of additional quanta, is treated. *N. Rosen* (Haifa).

Edwards, S. F. On the divergence of the perturbation method in field theory. Phil. Mag. (7) **46** (1955), 569–570.

In a previous note [Phil. Mag. (7) **45** (1955), 758–761; MR **16**, 101], the author treated the question of the convergence of the perturbation expansion of the one-nucleon Green's function in quantum field theory neglecting vacuum polarization effects and assuming a Feynman cut-off function to converge the integrals. It was found there that the series is asymptotic independent of whether the nucleon is treated as a fermion or boson particle. In the present paper the same problem is considered including vacuum polarization effects. The results change in a startling fashion. If the nucleon is treated as a fermion its Green's function can be represented as the ratio of two convergent power series in the coupling constant while for a boson nucleon the Green's function is the ratio of two asymptotic series. Thus even in a cut-off field theory Bose-Bose interactions have a zero radius of convergence while Fermi-Bose interactions have a finite radius of convergence.

R. Arnowitt (Syracuse, N.Y.).

Heisenberg, Werner. Bemerkungen zur "neuen Tamm-Dancoff-Methode" in der Quantentheorie der Wellenfelder. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1956**, 27–36.

Consider a system with one single spinor field $\psi_\alpha(x)$. A state $|\phi\rangle$ can be characterized either with the aid of the set of functions $\tau(x_1, x_2, \cdots|y_1, y_2, \cdots)$ defined as

$$\tau(x_1, x_2, \cdots|y_1, y_2, \cdots)=\langle\phi|T\psi(x_1)\psi(x_2)\cdots\psi^+(y_1)\psi^+(y_2)\cdots|\Omega\rangle,$$

where $|\Omega\rangle$=the physical vacuum, or by the set $\delta(x, \cdots, y, \cdots)$ defined from the expansion

$$|\phi\rangle=\Big\{\int dx\sigma(x|)\,\psi(x)+\int dy\sigma(|y)\psi^+(y)+\int dx_1dx_2\sigma(x_1,\ x_2|)T\psi(x_1)\psi(x_2)+\cdots\Big\}|\Omega\rangle.$$

The two sets are in a certain way analogous to a covariant and a contravariant description of ordinary vectors as

the normalization integral of the state $|\phi\rangle$ can be written

$$\langle\phi|\phi\rangle=\int\sigma(x|)\tau(x)dx+\int\sigma(|y)\tau(|y)dy+\int\sigma(x_1, x_2|)\tau(x_1, x_2|)dx_1dx_2+\cdots.$$

To investigate the usefulness of these sets and, in particular, the error made in neglecting all functions with more than a given number of variables in them, the author studies the example of the anharmonic oscillator $\ddot{q}=-\omega^2q-\lambda q^3$. The following things are shown with the aid of explicit calculations: (i) If the anharmonic term λq^3 is small, the error made in neglecting all functions σ with more than n arguments is of the same order of magnitude as the error made in neglecting all functions τ with more than n arguments. (ii) If the term proportional to q is zero and if only two functions are used in each of the expansions above, the result agrees reasonably well with the result of a numerical integration of the exact equation.

These results are then considered as a sufficient motivation to use the same technique for the infinitely more complicated case of a spinor field coupled to itself in the following way

$$\gamma_\nu\frac{\partial\psi(x)}{\partial x_\nu}+l^2\psi(x)(\psi^+(x)\psi(x))=0.$$

The author has previously suggested that this equation can be used as a rough model for a system of elementary particles. The two first τ-functions in this model have been worked out earlier by the author with Kortel and Mitter [Z. Naturf. **10a** (1955), 425–446; MR **17**, 330] and the two first σ-functions are given in the present paper. Both calculations are done in an approximation which, among many other things, implies that all higher functions are neglected. The resulting normalization integral is claimed to be "a measure of the density of matter in the elementary particle" and studied in some detail. The corresponding density function is shown to drop off rather rapidly at infinity but to have a very strong singularity at the origin. It is suggested that this singularity will be removed if higher τ- and σ-functions are included in the calculation. *G. Källén.*

Gombás, P. Über eine Modifikation der Weizsäckerschen Korrektion im statistischen Atommodell. Ann. Physik (6) **18** (1956), 1–16.

It has been shown in earlier work by Sokolov [Ž. Eksper. Teoret. Fiz. **8** (1938), 365–376] and by the author [Acta Phys. Acad. Sci. Hungar. **3** (1953), 105–125] that the correction given by von Weizsäcker [Z. Physik **96** (1935), 431–458] to the electron kinetic energy in the Thomas-Fermi statistical model of the atom leads to numerical results which are too large. In this paper the difficulty is traced to the manner of inclusion of the radial component of momentum in both the Fermi and the v. Weizsäcker evaluations of the kinetic energy. This is corrected by a procedure which partitions momentum space into spherical shells. This method not only gives a correction to the mean radial energy, but also sets a positive lower bound on the magnitude of the momentum. The result is a substantial improvement in the calculated energies and electron distribution functions for atoms. The electron density is finite at the nucleus and diminishes exponentially at very large distances, as is the case for the wave mechanical theory.

E. L. Hill.

Abarenkov, I. V. On the Schrödinger coordinate function for an arbitrary state of an atom with several electrons. Vestnik Leningrad. Univ. **11** (1956), no. 10, 43–54. (Russian)

The dependence of the Schrödinger wave function of an N-electron system on the coordinates r_i, for a fixed configuration of the spins σ_i, is derived for a broad class of wave functions. This class consists of those antisymmetric wave functions Ψ expressible in terms of N one-electron orbitals φ_i by

$$\Psi=\sum_{\alpha_1\cdots\alpha_N}a(\alpha_1, \cdots, \alpha_N)\varphi_1(r_{\alpha_1})\cdots\varphi_N(r_{\alpha_N})X(\sigma_{\alpha_1}\cdots\sigma_{\alpha_N}),$$

where the a's are any coefficients and the X's eigenfunctions of the total spin S and its z-component. The φ's need not all be different. It is shown that when a given $(\frac{1}{2}N+S)$ spins are $+$, the remaining $-$, the form of Ψ is a sum of products of pairs of determinants constructed from the φ_i. *C. Herring* (Murray Hill, N.J.).

Petrašen', M. I.; Ivanova, A. V.; and Vol'f, G. An elementary method for taking account of the influence of the field of a crystal lattice on the one-electron S- and P-functions of an ion. Vestnik Leningrad. Univ. **11** (1956), no. 10, 29–38. (Russian)

It is assumed that a reasonable approximation to the self-consistent field solution for an ionic crystal can be obtained in terms of one-electron wave functions for each ion which obey a wave equation differing from that of the free ion only by inclusion of the Coulomb field of the surrounding ions, these being treated as point charges. The development of this Coulomb field in spherical harmonics is carried through explicitly for the rocksalt lattice, the analytical form of the expression changing each time the distance from the central ion passes through a value equal to the distance of one of the shells of neighbors. Applications to the polarizability and diamagnetic susceptibility of Li F are given. *C. Herring.*

Dyson, Freeman J. Scattering of mesons by a fixed scatterer. Phys. Rev. (2) **100** (1955), 344–348.

The meson-nucleon scattering equations of Chew and Low, [Phys. Rev. (2) **101** (1956), 1570–1579] are generalized to the case of mesons and scatterers having arbitrary angular momenta. The usual method of eliminating the magnetic quantum numbers is to use the fact that the total angular momentum and its z-component are constants of motion, and to couple the initial angular momenta of the meson and the scatterer. An alternative way to eliminate the magnetic quantum numbers is to couple together the initial and final angular momenta of the meson. The author studies both coupling schemes, and finds that the condition of unitarity of the S-matrix is simple only in the first scheme, while the condition of causality is simple only in the second scheme. The interlock between the two schemes gives the characteristic linking of J-values in Chew-Low equations, the linkage coefficients being Racah coefficients. *A. Salam.*

★**Kockel, Bernhard. Darstellungstheoretische Behandlung einfacher wellenmechanischer Probleme.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1955. 232 pp. (VI Tafeln). DM 18.50.

According to the foreword, this little book originated from lectures given by the author in the desire to come to a good understanding of van der Waerden's classical book on group-theoretical methods in quantum mechanics, and a fairly accurate description of its scope can be given

by saying that the author tries to explain in detail to the non-mathematically minded physicists part of what they could learn from van der Waerden's monograph. The quantum mechanics used is most elementary: the non-relativistic Schrödinger equation and the formula for dipole radiation of systems of charged particles. The mathematics too assumes very little known: both matrix calculus and group theory are exposed in detail and on an elementary level. Every point is carefully illustrated by means of simple examples. The only group for which the representation theory is presented is the three-dimensional rotation group. The consequences of the exclusion principle are discussed. *L. Van Hove* (Utrecht).

Avrorin, E. N.; and Fradkin, E. S. Renormalizability of pseudoscalar meson theory with pseudovector coupling. Ž. Eksper. Teoret. Fiz. **30** (1956), 756–760, supplement to **30**, no. 4, 7. (Russian. English summary)

It was conjectured by N. Hu [Phys. Rev. (2) **80** (1950), 1109–1110; MR **12**, 573] that the non-renormalizable infinities appearing in the theory of pseudo-scalar mesons with pseudo-vector coupling are the result of an incorrect application of perturbation theory, and that the use of the exact propagation function for the meson would reduce the infinities to renormalizable ones. According to Hu, the exact propagator ought to fall off for large meson momentum, k, faster than k^{-4}, and this would lead in perturbation theory to a series in which the first radiative correction would increase as k^4 for large k. The present authors show that previous calculations of the first radiative corrections to the meson propagator are incorrect because of an uncritical use of regularization in the evaluation of ambiguous integrals. They point out that the theory has an analogue in the charge conservation law of electrodynamics (which says that the divergence of the pseudo-vector covariant of the spinor field is proportional to the pseudo-scalar covariant), which can be used to resolve ambiguities in the handling of infinite quantities. Using it they find that the first radiative correction to the meson propagator goes as k^2, not k^4. This result is also obtained by using a limiting process analogous to that employed by Valatin in electrodynamics [Proc. Roy. Soc. London. Ser. A. **226** (1954), 254–265; MR **16**, 319]. The authors conclude that if pseudo scalar meson theory with pseudovector coupling is renormalizable at all, the convergence must arise from the use of the exact vertex operator.

A. S. Wightman (Princeton, N.J.).

Faĭnberg, V. Ya. Non-linear equations in quantum field theory. Ž. Eksper. Teoret. Fiz. **30** (1956), 608–609. (Russian)

The connection of the approximate non-linear equation for the S matrix element of meson-nucleon scattering derived in a special theory by F. Low [Phys. Rev. (2) **97** (1955), 1392–1398] with the exact non-linear relation for vacuum expectation values derived by Lehmann, Symanzik, and Zimmermann [Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219] is pointed out.

A. S. Wightman (Princeton, N.J.).

Vrkljan, Vladimir S. Die de Broglieschе Theorie der Partikeln mit dem Spin 1 und das Theorem von Ehrenfest. Bull. Internat. Acad. Yougoslave Sci. Beaux-Arts (N.S.) **6** (1952), 85–87.

A translation of the article reviewed in MR **15**, 588.

Vrkljan, Vladimir S. Über das magnetische Moment des Mesons mit dem Spin 1. Bull. Internat. Acad. Yougoslave Sci. Beaux-Arts (N.S.) **6** (1952), 89–90.

A translation of the article reviewed in MR **15**, 588.

Jakobi, Georges; et Lochak, Georges. Introduction des paramètres relativistes de Cayley-Klein dans la représentation hydrodynamique de l'équation de Dirac. C. R. Acad. Sci. Paris **243** (1956), 234–237.

The authors express an arbitrary four component spinor in terms of eight real parameters, two associated with the values of the scalar and pseudo-scalar determined by the spinor and the remaining six determined by the independent components of the current vector and the pseudo-vector associated with the spinor. The Dirac Hamiltonian is then written in terms of the eight parameters listed above. The non-relativistic limit of the Hamiltonian is also given and shown to tend to that of the Pauli theory of a spinning electron. *A. H. Taub.*

Jakobi, Georges; et Lochak, Georges. Decomposition en paramètres de Clebsch de l'impulsion de Dirac et interprétation physique de l'invariance de jauge des équations de la mécanique ondulatoire. C. R. Acad. Sci. Paris **243** (1956), 357–360.

The authors express a combination of the Gordon current vector and a pseudo vector defined in an analogous manner which they call the Proca vector in terms of the parameters used in the paper reviewed above to represent the components of a four component spinor. The resulting expression is interpreted in terms of the Clebsch decomposition of an arbitrary vector in terms of gradients of scalars. *A. H. Taub* (Urbana, Ill.).

See also: Freĭman, p. 112; Fubini, p. 135; Melvin and Edwards, p. 164; Galasiewicz, p. 170; Dolginov, p. 176; Szamosi, p. 176.

Relativity

Dolginov, A. Z. Relativistic spherical functions. Ž. Eksper. Teoret. Fiz. **30** (1956), 746–755, supplement to **30**, no. 4, 6. (Russian. English summary)

Finite-dimensional representations of the rotation group in four-dimensional Euclidean and pseudo-Euclidean spaces are considered. The matrices of the irreducible representations of the group are determined. The Clebsch-Jordan coefficients are obtained. Relativistic spherical scalar, spinor and vector functions are constructed. As an example, the use of spherical functions in solving a relativistic equation of the Bethe-Salpeter type is discussed. *N. Rosen* (Haifa).

Szamosi, G. Die relativistische Bewegung des Massenpunktes bei einer allgemeinen Kraftannahme. Acta Phys. Acad. Sci. Hungar. **5** (1956), 463–469. (Russian summary)

The motion of a particle is considered in the framework of the classical special relativity theory, under the assumption that it is acted on by a general force which can change its rest-mass. In the case of a force which is attractive at large distances from the origin, it may happen that at smaller distances the rest-mass changes sign, and the particle experiences a repulsion. The corresponding wave equation in quantum mechanics is briefly discussed. *N. Rosen* (Haifa).

Moon, Parry; and Spencer, Domina Eberle. On the establishment of a universal time. Philos. Sci. **23** (1956), 216–229.

The problem is to synchronize the clocks of observers in relative motion in a Euclidean space who can interchange light-signals. It is assumed that the velocity of light can depend on the velocity of its source. The relative velocity of two observers is defined only by the statement that it "may be determined by ordinary astronomical procedures or ··· by Doppler effect". In the solar system, to which the authors' ideas are intended to apply, both these procedures pre-suppose the existence of the absolute (universal) time of Newtonian mechanics. Another postulate used, but not stated, is that the rate of a clock is the same whether the clock is regarded as being in motion or at rest. {In view of this, it is not surprising that the authors do not recover the equivalent of special relativity. The proposed theory will synchronize clocks if it is assumed that the velocity of light is variable and also if a universal time is pre-supposed; it can hardly be said to establish such a time from first principles even with the degree of success achieved, for example, by Milne's kinematical relativity.} *G. C. McVittie* (Urbana, Ill.).

Fierz, M. Über die physikalische Deutung der erweiterten Gravitationstheorie P. Jordans. Helv. Phys. Acta **29** (1956), 128–134.

By choosing appropriate integrals to vary, it is proved that the postulate that uncharged mass-points trace out geodesics in Jordan's theory is equivalent to the postulate that the Compton wave-length of elementary particles is constant. The gravitation constant $\varkappa$ is defined as the ratio of gravitational to inertial mass. It is shown that the dielectric constant of a vacuum is equal to $\varkappa^{1+1/\eta}$ where η is not zero. *G. C. McVittie* (Urbana, Ill.).

Taub, A. H. Isentropic hydrodynamics in plane symmetric space-times. Phys. Rev. (2) **103** (1956), 454–467.

The author has derived elsewhere a form for the energy-tensor of general relativity that has the advantage of displaying specifically the function corresponding to the internal energy of the fluid. It is governed by five differential relations that give the conservation of mass, energy and momentum. This energy-tensor is used to study solutions of Einstein's equations for space-times with plane symmetry, the motion of the fluid being irrotational and isentropic and subject to its own gravitational field as well as to the hydrodynamical forces. It is proved that a co-moving coordinate system can be found that has an analogy with the Lagrange coordinates of classical hydrodynamics. The method is illustrated by considering static solutions and solutions which depend on the time alone. The boundary conditions of different classes are discussed. When the fluid is incompressible it is proved that such a fluid cannot bound a vacuum unless the boundary condition of the continuity of the derivatives of the metric tensor is violated. The time-dependent solutions for the incompressible fluid are worked out in detail. *G. C. McVittie* (Urbana, Ill.).

Cahen, Michel. Conditions d'intégrabilité du champ électromagnétique singulier. C. R. Acad. Sci. Paris **243** (1956), 737–740.

Let us have in space-time an electromagnetic field and a gravitational field, solutions of Maxwell's equations and the Einstein equation respectively. Furthermore, suppose that the two invariants of the electromagnetic field tensor are zero. Then the electromagnetic field tensor specifies two families of orthogonal surfaces, both tangent to the light cone. [For the corresponding theorems in flat spacetime and for their physical interpretation, see J. L. Synge, Relativity: the special theory, North Holland Publ. Co., Amsterdam, 1956, Ch. IX; MR **17**, 1013]. *N. L. Balazs* (Chicago, Ill.).

Bertotti, B. Gravitational motion and Hamilton's principle. Nuovo Cimento (10) **3** (1956), 655–657.

The author derives a variational equation satisfied by the stress energy tensor generating a gravitational field, when the coordinate system is changed. He then identifies this stress energy with that of a test particle and claims to prove that as a consequence of the previous variational principle the test particle moves along a geodesic in the space and has a constant rest mass. {Since a test particle is supposed to have no influence on the gravitational field the reviewer cannot see the justification for identifying the stress energy tensor of the test particle with that tensor which "creates" the gravitational field.} *A. H. Taub* (Urbana, Ill.).

Narlikar, V. V.; and Rao, B. R. The problem of motion in general relativity. Proc. Nat. Inst. Sci. India. Part A. **21** (1955), 416–427 (1956).

The authors review the work of Einstein and Infeld [Canad. J. Math. **1** (1949), 209–241; MR **11**, 59] on the derivation of the equations of motion from the field equations. It is pointed out that the masses and coordinates associated with the singularities representing the particles should also be considered as functions of the expansion parameter λ as is not the case of the Einstein-Infeld treatment. This paper contains a method of approximation which treats the masses and coordinates as functions of λ and determines the equations of motion of the singularities from the field equations. It is shown that the equations of motion to sixth order are those given by the Einstein-Infeld method and solved by Robertson [Ann. of Math. (2) **39** (1938), 101–104]. However in higher approximations the masses are functions of the time. *A. H. Taub* (Urbana, Ill.).

Narlikar, V. V.; and Rao, B. R. The equations of motion of particles in the unified field theory of Einstein (1953). Proc. Nat. Inst. Sci. India. Part A. **21** (1955), 409–415 (1956).

The authors apply the method of Einstein and Infeld to the determination of the approximate equations of motion of a charged particle from the 1953 generalized Einstein theory of gravitation. The approximation is carried out to third order in λ, the expansion parameter. By choosing an "electromagnetic potential" which is composed of a term linear in r and a term linear in $1/r$ they obtain equations of motion with terms corresponding to Coulomb's law of force. Callaway [Phys. Rev. (2) **92** (1953); 1567–1570; MR **15**, 564] has obtained equations of motion without such terms by using a different type of "electromagnetic" potential. *A. H. Taub* (Urbana, Ill.).

Tonnelat, M. A. Les équations approchées de la théorie du champ unifié d'Einstein-Schrödinger. Nuovo Cimento (10) **3** (1956), 902–920.

The author has previously determined the components of the affine connection of the generalized Einstein theory of relativity as functions of the components of the asym-

metric $g_{\mu\nu}$ [J. Phys. Radium (8) **12** (1951), 81–88; MR **13**, 79]. These results are used to obtain approximate equations satisfied by the gravitational (symmetric part of $g_{\mu\nu}$) and electromagnetic (anti-symmetric part of $g_{\mu\nu}$) tensors. The approximations made involve expanding the anti-symmetric part of $g_{\mu\nu}$ in terms of powers of a parameter and keeping third order terms in this parameter.

A. H. Taub (Urbana, Ill.).

Husain, S. I.; and Mishra, R. S. Projective change of affine connections in Einstein's unified field. Tensor (N.S.) **6** (1956), 26–31.

Dans la théorie unitaire non symétrique d'Einstein on appelle changement projectif de connexion le changement préservant les géodésiques:

$$*\Gamma^i_{jk}=\Gamma^i_{jk}+2\delta_j{}^i\lambda_k$$

[cf. Einstein, The meaning of relativity, 4ième éd., Princeton, 1953; MR **14**, 805; Lichnerowicz, Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955, p. 248; MR **17**, 199]. Les auteurs montrent que les tenseurs

$$I^i_{jkl}=R^i_{jkl}+\delta_j{}^iR_{kl}\ (R_{kl}=R^i_{kil})$$

$$F^i_{jkl}=R^i_{jkl}-\tfrac{1}{4}\delta_j{}^i\Lambda_{kl}\ (\Lambda_{kl}=R^i_{kil})$$

qui sont des invariants projectifs, vérifient dans la théorie unitaire d'Einstein, les équations:

$$I^i_{jkl,m}+I^i_{jlm,k}+I^i_{jmk,l}\equiv 0,$$

$$F_{jk,l}+F_{kl,j}+F_{lj,k}\equiv 0.$$

Ils expriment ensuite les identités de Bianchi de la théorie unitaire [cf. Einstein, Canad. J. Math. **2** (1950), 120–128; MR **11**, 548; Lichnerowicz, loc. cit., pp. 270–273] à l'aide de ces tenseurs. *Y. Fourès-Bruhat* (Marseille).

Narlikar, V. V. The unified field theory. Science and Culture **21** (1956), 495–502.

This is an expository article in which the author describes the general theory of relativity; some "imperfections" in this theory are pointed out and a general description is given of Einstein's recent generalized (unified field) theory of relativity. *A. H. Taub.*

See also: Wigner, p. 173.

Astronomy

Have

★**Siegel, Carl Ludwig. Vorlesungen über Himmelsmechanik.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. ix+212 pp. DM 29.80.

This book is divided into three chapters headed: The Three Body Problem, Periodic Solutions, and The Stability Problem. After a standard variational approach to dynamics followed by a reproduction of the proof of Cauchy's existence theorem for analytic solutions of differential equations there is given a derivation of the known integrals for the n-body problem. This is all preliminary to the climax of the first chapter: A demonstration of the theorem of Sundman that in the three body problem a collision of two of the particles represents merely a removable singularity. A regularizing parameter exists such that the coordinates of the particles (as well as the time, t) are analytic functions of this parameter, even through a binary collision. Apart from Birkhoff's book [Dynamical systems, Amer. Math. Soc. Colloq. Publ., v. 9, New York, 1927] where a brief sketch of the proof of this theorem is given and Wintner's [The analytical foundations of celestial mechanics, Princeton, 1941; MR **3**, 215] where it is treated extensively, considerations of this type are not to be found in the usual texts. The chapter on periodic solutions first gives an account of Lagrange's solutions with an extension to periodic solutions in the neighborhood of these. Hill's lunar theory is presented but with a somewhat different method for handling the non-linear differential equations than Hill used. Two existence-type methods, due in essence to Poincaré, close the chapter. The first method treats the possibility of determining periodic solutions for a set of initial conditions neighboring to those for a known periodic solution. The second method is based on the Poincaré-Birkhoff fixed point theorem. Both methods are applied to the restricted three body problem. The topological considerations of Poincaré, Liapounoff, and their followers concerning the stability problem occupy the last chapter. The final section is devoted to the ergodic theorem and its application.

The book is well written and fully detailed. The subject matter of the second and third chapters makes it a fitting complement to the above-mentioned book by Wintner.

R. G. Langebartel (Urbana, Ill.).

Skabickiĭ, I. N. Application of the kinetic equations to stellar systems. Leningrad. Gos. Univ. Uč. Zap. 136. Ser. Mat. Nauk **22** (1950), 10–32. (Russian)

Non-steady spherically symmetric systems are considered taking into account the effect of collisions. To a first approximation the spatial density distribution does not change and the velocity distribution approaches a Maxwellian. For a ten parsec radius system containing 10^6 stars the time of relaxation is 6×10^9 years.

R. G. Langebartel (Urbana, Ill.).

Sobolev, V. V. Diffusion of radiation with redistribution of frequencies. II. Vestnik Leningrad. Univ. **10** (1955), no. 11, 99–111. (Russian)

In his earlier paper [same Vestnik **10** (1955), no. 5, 85–100; MR **17**, 1142] the author derives formulae for the radiation field in a diffusive material based on a study of the integral equation for the coefficient of radiation. In the present paper he obtains formulae for the radiation as it leaves the material based on the equation for the probability of quantum departures from the material. He indicates how results could be applied to the construction of spectral line contours, the determination of the pressure of light at the material's boundary and of the number of atoms in the excited state.

R. G. Langebartel (Urbana, Ill.).

Kippenhahn, Rudolf. Untersuchungen über rotierende Sterne. I. Die Theorie nullter Ordnung. Z. Astrophys. **38** (1955), 166–189.

Extremely slowly rotating stars are considered such that all quantities of at least the first order in the quantity $\bar{\omega}^2/2\pi G\bar{\rho}$ are neglected, where $\bar{\omega}$ is the mean angular velocity, G the gravitation constant, and $\bar{\rho}$ the mean density. The essential functions involved (density, temperature, etc.) are assumed to change exponentially with the time, t, thereby making the equations of motion for the standard model a characteristic value problem. The first characteristic function for a class B4 star is obtained numerically which gives an angular velocity at the center nine times that at the surface. For very large t

the first characteristic function is a good approximation to the solution, but it appears that no observed stars have had a sufficiently long lifetime for this approximation to hold. *R. G. Langebartel* (Urbana, Ill.).

Radzievskiĭ, V. V. Tensor of radiation pressure. Astr. Ž. 33 (1956), 129–136. (Russian. English summary)

It is shown by means of an example that electromagnetic radiation pressure must sometimes satisfy physical conditions which preclude its being a tensor. In general the radiation pressure at a point is a tensor only for those cartesian coordinate systems having a common origin at the point and with respect to which the directions of radiation propagation lie in the first octant. Consequences of this result for the calculation of the radiation pressure of stars are discussed, the most important being that at a point lying on or near a closed radiating surface or in its interior, radiation pressure is not a tensor.

R. N. Goss (San Diego, Calif.).

Agekyan, T. A. On the dynamics of stellar passages through a cloud of meteoritic material. Leningrad. Gos. Univ. Uč. Zap. **136**. Ser. Mat. Nauk **22** (1950), 33–46. (Russian)

The author determines the rate of accretion, neglecting radiation pressure, of the meteoritic material due to inelastic collisions of particles on the symmetric axis behind the star. He finds that to explain the origin of the solar system in this way demands a high density particle cloud and a low velocity of the cloud relative to the stars. [Cf. the theory of Hoyle and Lyttleton, Proc. Cambridge Philos. Soc. **35** (1939), 405–415.]

R. G. Langebartel.

★ **Kobrin, M. M. On the phase radar method of measuring the distance to the moon.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translated from Trudy, 5th Conference on Questions of Cosmogony, 'Radio Astronomy', Akad. Nauk Press, Moscow 1956, pp. 141–146.

Hoppe, J. Untersuchungen zur physikalischen Theorie der Meteore. I. Astr. Nachr. **283** (1956), 95–108.

By building an energy and impulse transfer theory on the direct collisions of air particles with a non-fireball type meteor the author is able to derive relations for the velocity and mass decay of the meteor upon penetration of the atmosphere. He considers spherical, cylindrical, and conical meteors. *R. G. Langebartel.*

Bracewell, R. N. Strip integration in radio astronomy. Austral. J. Phys. **9** (1956), 198–217.

See also: Piddington, p. 170.

Geophysics

★ **Scheffers, Georg. Wie findet und zeichnet man Gradnetze von Land- und Sternkarten? Zweite Auflage, verbessert und erweitert von Karl Strubecker.** Mathematisch-physikalische Bibliothek, Reihe I, 85/86. B. G. Teubner, Verlagsgesellschaft, Stuttgart, 1956. 114 pp.

The original (1934) edition was an elementary textbook on cartography emphasizing the construction of atlas-type projections from a geometric point of view, and using no mathematics beyond the high school level. Within these restrictions, a large number of projections of the sphere onto the plane are quite adequately discussed. The principal change in this new edition is the inclusion of a 16 page appendix covering briefly the elements of differential geometry, and in particular the analytic derivations of the mercator, stereographic and equal-area projections of the sphere onto the plane. *B. Chovitz.*

See also: Stovas, p. 145; Bjerhammar, p. 153; Gifford, p. 167.

OTHER APPLICATIONS

Games, Economics

Dulmage, A. L.; and Peck, J. E. L. Certain infinite zero-sum two-person games. Canad. J. Math. **8** (1956), 412–416.

The game (I, J, K) consists of two arbitrary sets I and J and a real function on the product set $I \times J$. Let $\xi=\{x_i | i \in I\}$ be a real vector such that $\sum x_i=1$ and all $x_i \geqq 0$. This is used as a mixed strategy for the maximising player. Similarly $\eta=\{y_j\}$ is used as a mixed strategy for the minimising player. Let $K(\xi, \eta)=\sum_{i,j} K(i, j)x_i y_j$. By $\inf_\eta K(\xi, \eta)$ is meant the infimum over all those η for which $K(\xi, \eta)$ exists. Let

$$\bar{v}_{IJ}=\sup_{\xi} \inf_{\eta} K(\xi, \eta) \text{ and } \bar{v}_{IJ}=\inf_{\eta} \sup_{\xi} K(\xi, \eta).$$

If $K(\xi, \eta)$ exists for all ξ, η, then $\bar{v}_{IJ} \leqq \bar{v}_{IJ}$. Games for which $\bar{v}_{IJ} \leqq \bar{v}_{IJ}$ are called admissible. For any admissible game and finite subset n of J, $\bar{v}_{IJ} \leqq \bar{v}_{In} \leqq \bar{v}_{IJ}$. Let M be the collection of all finite subsets of I. If f is a real valued function on I, $\lim_i f(i)=A$ means that for each $\varepsilon>0$ there is an $m \in M$ such that for $i \notin m$, $|f(i)-A|<\varepsilon$. If g is a real valued function on M, then $\lim_{m \in M} g(m)=A$ means that for each $\varepsilon>0$ there is an $m' \in M$ such that for $m \supset m'$, $|g(m)-A|<\varepsilon$. Theorem: If (I, J, K) admissible and for each $j \in J$ there is a real number L_j such that $\inf_i K(i, j)=L_j=\lim_i K(i, j)$ the game has a value $v=\lim_{n \in N} \bar{v}_{In}$, $v<+\infty$ and the maximising player has an optimal strategy. This is applied to infinite hide and seek games. *S. Sherman* (Philadelphia, Pa.).

Basmann, R. L. A theory of demand with variable consumer preferences. Econometrica **24** (1956), 47–58.

This paper explores some aspects of the theory of consumer demand under variable preferences. A number of "taste" parameters are introduced in the preference function. Let a be such a parameter. Differentiating the optimum consumption vector x_i $(i=1, 2, \cdots, n)$ with respect to a, Tintner [Metroecon. **4** (1952), 1–4] gave the explicit expression of the derivatives dx_i/da. Rewriting the formula in terms of the elasticity $a/x_i \cdot dx_i/da$, Basmann finds this elasticity simply to equal a weighted average of all income-compensated price elasticities of the demand of x_i, the weights being given by the elasticities of the marginal rates of substitution with respect to a. Basmann then derives an analogue for a change of a to Leontief-Hicks' theorem on composite commodities. The new

theorem says that a group of goods, all of whose marginal rates of substitution change in equal proportion, can be treated as a single commodity. {The proof is, however, not complete, since the integrability is not established.}
S. A. O. Thore (Stockholm).

Bowman, Mary Jean. The analysis of inequality patterns: a methodological contribution. Metron **18** (1956), no. 1–2, 189–206.

Benson, F. Probability and operational research. Research **9** (1956), 329–334.
"In this article, the author has attempted to show by means of examples some of the ways in which probability models can be used to solve executive problems in industry. Many other industrial problems are amenable to this approach which is one of the main disciplines that make up the operational research worker's stock-in-trade." (Author's summary.) *J. Kiefer.*

Lieberman, Irving J. A mathematical model for integrated business systems. Management Sci. **2** (1956), 327–336.
Suggests using a sequence of one-directional incidence matrices to trace the flow of information through an organization. *R. Solow* (Cambridge, Mass.).

Zentler, A. P.; and Ryde, Dorothy. An optimum geographical distribution of publicity expenditure in a private organisation. Management Sci. **2** (1956), 337–352.
This paper attempts the solution to a practical problem. Given an international organization with branches in n countries and concerned with publicity for a commodity X — in competition with a substitute X' — how should such an organization allocate its expenditure among them so as to get the maximum overall return. (Author's summary.) *R. Solow* (Cambridge, Mass.).

Naddor, Eliezer. Some models of inventory and an application. Management Sci. **2** (1956), 299–312.
Discusses, with examples, several standard models for determining optimal inventory levels. The models differ according to whether time is taken as discrete or continuous, and whether demand occurs evenly throughout each planning period or instantaneously at the beginning.
R. Solow (Cambridge, Mass.).

See also: Rybarz, p. 160.

Biology and Sociology

Whittle, P. The estimation of age-specific infection rates from a curve of relative infection. Biometrics **12** (1956), 154–162.
A population of N individuals, healthy at time $t=0$, is subject to a natural death rate $\alpha(t)$, an infection rate $\lambda(t)$, and a death rate $\beta(s, t)$ from infection contracted at time $t-s$. A general formula is obtained for the proportion of infected individuals amongst those alive at time t, and two special cases are considered for the duration of sickness: (a) fixed, (b) exponentially distributed. In an application to an infection of wild rabbits (a) appears to be unsatisfactory, but on assumption (b) some plausible bounds are obtained for $\lambda(t)$. *P. Armitage.*

Lewontin, R. C.; and Prout, Timothy. Estimation of the number of different classes in a population. Biometrics **12** (1956), 211–223.
An experiment in genetics whose object is the estimation of the number of genes capable of lethal mutation can be described as follows. A total of N objects is randomly distributed among n cells, the probability of any particular arrangement being n^{-N}. The number n of cells is to be estimated; what can be observed for each pair (i, j) is whether or not the ith and jth objects fall into the same cell. From the $\frac{1}{2}N(N-1)$ comparisons one can calculate the number k of occupied cells, and this statistic k is sufficient for the estimation of n. The authors examine the maximum likelihood estimate $\hat{n}$ of n and give an approximation to its sampling variance. They also examine a cruder estimate $\tilde{n}$ based on the division of $2N'$ such objects into two equal paired groups and utilizing only the N' comparisons between pairs. The efficiency of $\tilde{n}$ (in relation to $\hat{n}$) is studied; the results depend on the relation which must connect N with N' if the two experiments are to involve equal amounts of experimental effort. *D. G. Kendall* (Oxford).

Caranti, Elio. Su un procedimento approssimato per la determinazione del numero medio dei figli per matrimonio. Metron **18** (1956), no. 1–2, 207–218.

See also: Ledley, p. 155; Graybill, Martin and Godfrey, p. 160; Womersley, p. 166.

Information and Communication

★ **Kharkevich, A. A.; and Blokh, E. L. On the limiting output of a communication system.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 9 pp.
Translated from Radiotehnika **10**, No. 2, 1955, pp. 14–20.

★ **Kharkevich, A. A. Outline of general communication theory.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.
Translated from Gosudar. Izdat. Tekh.-Teor. Lit., Moskva, 1955.

★ **Vasil'ev, A. M. Geometric derivation of the formula for the communication-with-noise capacity of a channel with a special receiver.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 4 pp.
Translated from Radiotehnika **11**, no. 2, 1956, pp. 77–79.

Middleton, David; and van Meter, David. Detection and extraction of signals in noise from the point of view of statistical decision theory. I. J. Soc. Indust. Appl. Math. **3** (1955), 192–253.
The paper gives an exhaustive exposition of the formulation of a class of problems of communication and reception in the presence of noise as problems of statistical decision theory. Particular attention is paid to the differences between the various decision criteria, both from the abstract point of view, and by specific illustrations of the cost and loss involved.
The analytical work includes examples of radar

detection, and the development of weak-signal theory based on formal power series.

The paper includes an appendix listing some pertinent theorems of Wald, and an extensive bibliography.

E. Reich (Minneapolis, Minn.).

Schober, H. Informationstheorie in Optik und Fernsehen. Optik 13 (1956), 350–364.

See also: Middleton and van Meter, p. 180; Lieberman, p. 180.

Control Systems

★**Letov, A. M. Stability of nonlinear regulating systems.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

A translation of the table of contents and the foreword of the book reviewed in MR **17**, 487.

Meerov, M. V. On the autonomous character of multiply connected systems which are stable under unlimited increase of steady accuracy. Avtomat. i Telemeh. **17** (1956), 410–424. (Russian)

Nadler, M. On the utilization of operational calculus in the investigation of transition processes in servomechanisms that have an amplifier with saturation. Avtomat. i Telemeh. **17** (1956), 467–470. (Russian)

Cypkin, Ya. Z. Calculation of transients in nonlinear control systems. Avtomat. i Telemeh. **17** (1956), 500–512, appendix to no. 6, 3. (Russian. English summary)

The paper deals with calculation of transients in intermittent or sampling control systems where controlling pulse duration is proportional to the error. When the error deviations are finite these systems are essentially nonlinear.

As an example the author examines an intermittent control system with internal feedback.

Author's summary.

Kraus, Günther. Ein Beitrag zur Bemessung von automatischen Pilotreglern. Arch. Elek. Übertr. **10** (1956), 175–187.

MISCELLANEOUS

Dyck, Martin. Goethe's views on pure mathematics. Germanic Review **31** (1956), 49–69.

Lebesgue, Henri. De l'arithmétique à l'algèbre et à l'analyse mathématique. Enseignement Math. (2) **2** (1956), 49–60.

„ Ce manuscrit, retrouvé dans les papiers de M. Lebesgue, était destiné à servir de préface à un livre de l'analyse qui semble n'avoir pas été rédigé".

★**Pöschl, Klaus. Mathematische Methoden in der Hochfrequenztechnik.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. viii+331 pp. DM 36.00.

The first two-thirds of this book is a compilation of mathematical methods and formulas whose usefulness is not restricted to the high frequency domain. It contains chapters on scalar and vector fields, determinants and matrices, complex variables, methods of the theory of functions, Fourier series and integrals, fundamental concepts of statistics, linear differential equations of the second order, special functions (including among the more usual types the Tchebycheff polynomials and the gamma function), and an introduction to the solution of boundary value problems by approximation methods. While most of this material may be found in other compendia, notably Madelung [Die mathematischen Hilfsmittel des Physikers, 3rd ed., Springer, Berlin, 1936], Pöschl's book differs in the method of approach and the importance attached to the various topics. Thus, being intended for engineers, it devotes a whole chapter to Fourier series and integrals. The presentation is quite remarkable for its extreme meticulousness, particularly in the definitions. All procedures are explained in detail, and little knowledge is presupposed on the part of the reader. Another advantage, of special value to a student, is the fact that the use of the formulas is illustrated on a large number of concrete examples. The remaining portion of the book is concerned with specific aspects of high frequency. After a general chapter on Maxwell's equations, it contains chapters on cavity resonators, wave guides, radiating fields, and the motion of electrons. While the treatment is of necessity highly condensed, it is nevertheless of outstanding clarity. The book can readily be used for occasional reference. It is only to be hoped that the language barrier will not discourage the wide dissemination which this book deserves. *J. E. Rosenthal.*

★**Miller, Kenneth S. Engineering mathematics.** Rinehart & Co., Inc., New York, 1956. xii+417 pp. $6.50.

The general theory of linear differential equations and the practical technique (Frobenius method) are covered in Chapter Three; determinants and matrices are in Chapter One; network theory is in Chapter Six. In Chapter Two we cover a variety of special subjects connected with integration. Fourier series and integrals are treated in Chapter Four, with some applications not usually found in texts on advanced calculus (for example, correlation functions and the Wiener-Khintchine relations), and also the Laplace transform. Finally, Chapter Seven deals with engineering probability with emphasis on functions of random variables. The book may be used for the first year of graduate work, or, with the recent trend of including more mathematics in the undergraduate curriculum in engineering schools in junior and senior courses. *From the author's preface.*

★**Ringleb, F. Mathematische Formelsammlung.** Sammlung Göschen, Band 51/51a. Sechste, erweiterte Auflage. Walter De Gruyter & Co., Berlin, 1956. 278 pp. DM 4.80.

The fifth edition was reviewed in MR **11**, 423.

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